MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Mechanical Engineering

2.004 Dynamics and Control II
Fall 2007

Quiz 1

Friday, October 5, 2007

PLEASE DO NOT TURN OVER UNTIL EXAM STARTS

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Friday, October 5, 2007
Duration: 50min (11:05–11:55am)

A DC motor is connected to an op–amp circuit in cascade as shown in Figure 1. The op–amp circuit subsystem is shown in Figure 2; the input to the op–amp is a voltage source (battery) $v_i(t)$, the output is the voltage $v_s(t)$, and the transfer function of this subsystem is $G_1(s)$. The DC motor subsystem is shown in Figure 3; the input to the DC motor is the op–amp's output $v_s(t)$, the output is the angular velocity $\omega(t)$ of a shaft connected to the motor, and the transfer function of this subsystem is $G_2(s)$. The DC motor subsystem is not loading the op–amp circuit subsystem. We will analyze the behavior of each subsystem independently, and that of the overall cascaded system.

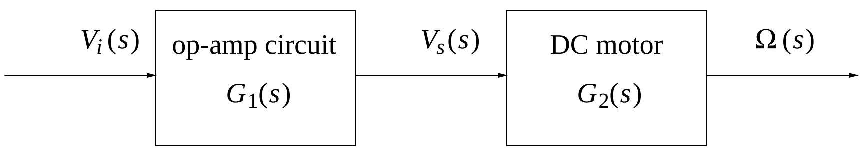


Figure 1: Cascade of an op–amp circuit subsystem and a DC motor subsystem.

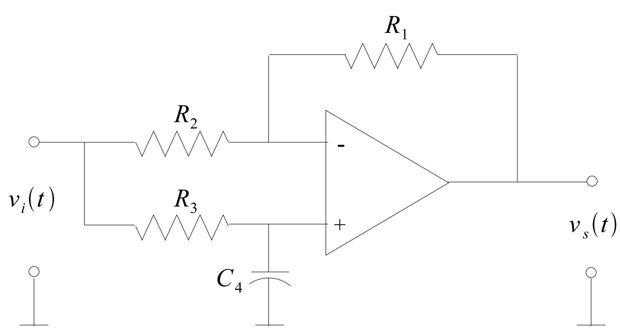


Figure 2: The op–amp circuit subsystem.

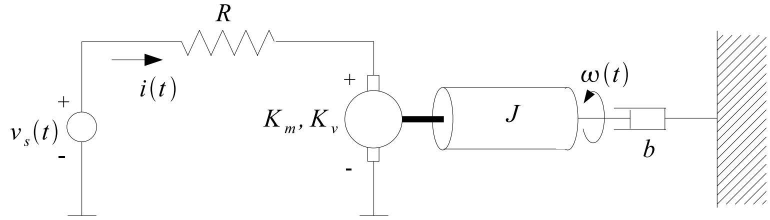


Figure 3: The DC motor subsystem.

Additional data for the op–amp circuit subsystem are: the resistors $R_1 = 100\Omega$, $R_2 = 100\Omega$ and $R_3 = 100\,\text{k}\Omega$, and the capacitor $C_4 = 1\mu F$.

Additional data for the DC motor subsystem are: motor resistance $R = 1\Omega$; the torque constant $K_m = 1\,\mathrm{N}\cdot\mathrm{m}/\,\mathrm{A}$ and back–emf constant $K_v = 1\,\mathrm{V}\cdot\mathrm{sec}/\,\mathrm{rad}$; the load shaft inertia $J = 1\,\mathrm{kg}\cdot\mathrm{m}^2$ and viscous friction coefficient $b = 1\,\mathrm{N}\cdot\mathrm{m}\cdot\mathrm{sec}/\,\mathrm{rad}$. The motor's own inertia and inductance are negligible.

1. **(25%)** Derive the transfer function $G_1(s)$ of the op–amp circuit subsystem. Locate the poles and zeros of $G_1(s)$ on the s–plane.

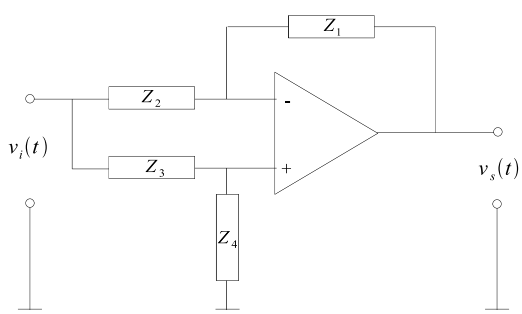
2. **(25%)** Derive the transfer function $G_2(s)$ of the DC motor circuit subsystem. Locate the poles and zeros of $G_2(s)$ on the s–plane.

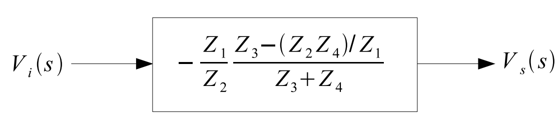
3. **(10%)** Derive the overall system transfer function (*i.e.*, with $V_i(s)$ as input and $\Omega(s)$ as output.) Locate the poles and zeros of the overall system transfer function on the s–plane.

4. **(25%)** Derive the time–domain response $\omega(t)$ when the input $v_i(t)$ is a step function of amplitude 1V (*i.e.*, the unit–step response.)

5. **(15%)** Sketch the result that you obtained in question 4. What is the slope $\mathrm{d}\omega(t)/\mathrm{d}t$ at $t = 0+$ (*i.e.*, an infinitesimal amount of time after the switch is turned on?) Is that observation consistent with any features of the transfer function that you derived in question 3?

Time–saving hint You may use the following result to simplify your calculations: The voltage–in/voltage–out transfer function of an op–amp connected in feedback configuration with four impedances Z_1, Z_2, Z_3, Z_4 is as shown below:





GOOD LUCK!

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1. (25%) The transfer function $G_1(s)$ of the op-amp subsystem.

Answer:

Let's denote $V_a(s)$ as the voltage at the terminals of the op–amp. (Note the voltage is the same on both terminals.) From the voltage divider rule, $V_a(s)$ can be written in the Laplace domain as

$$V_a(s) = \frac{1/C_4 s}{R_3 + 1/C_4 s} V_i(s).$$

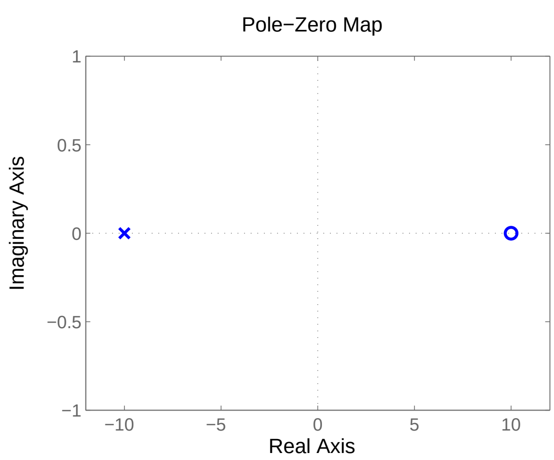
Since no current may flow through the op–amp terminals, (*i.e.*, the input resistance is infinite) the current I_2 flowing through R_2 is the same as the current through R_1.

$$I_2 = \frac{V_i(s) - V_a(s)}{R_2} = \frac{V_a(s) - V_s(s)}{R_1}.$$

Substituting V_a in the above equations, solving for V_s, and substituting the numerical values, we obtain

$$G_1(s) = \frac{V_s(s)}{V_i(s)} = -\frac{R_1}{R_2} \frac{s - R_2/\left(R_1 R_3 C_4\right)}{s + 1/\left(R_3 C_4\right)} = -\frac{s - 10}{s + 10}.$$

Alternatively, we can get the same answer by direct substitution in the formula given as "time–saving hint."



2. (25%) The transfer function $G_2(s)$ of the DC motor subsystem.

Answer:

From KVL (in the Laplace domain),

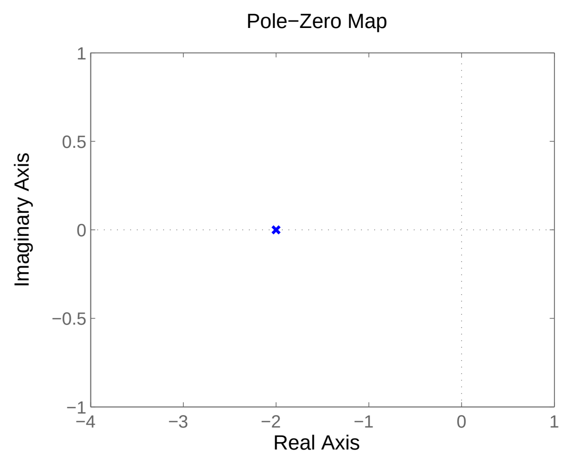
$$V_s(s) - I(s)R - K_v\Omega(s) = 0.$$

From torque balance at the motor shaft,

$$K_m I(s) = (Js + b)\Omega(s).$$

Rearranging the above two equations, and substituting the numerical values, we obtain

$$G_2(s) = \frac{\Omega(s)}{V_s(s)} = \frac{1}{s+2}.$$

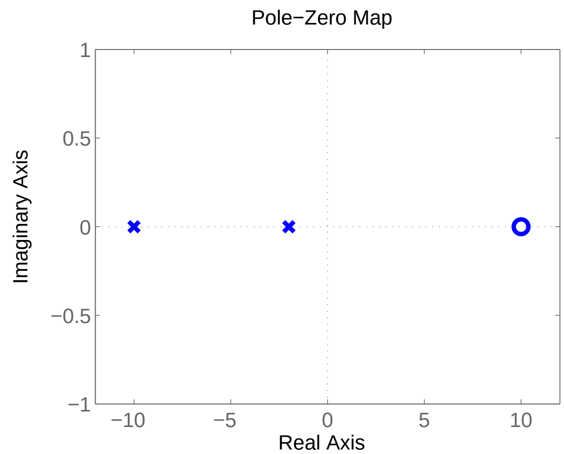


3. (10%) Overall system transfer function and location of poles and zeros.

Answer: Since the DC motor subsystem is not loading the op–amp circuit subsystem, the cascade transfer function is the product of the transfer functions of the two subsystems.

$$\begin{aligned}\frac{\Omega(s)}{V_i(s)} &= \frac{\Omega(s)}{V_s(s)} \times \frac{V_s(s)}{V_i(s)} = G_2(s) \cdot G_1(s) \\ &= -\frac{s-10}{(s+10)(s+2)}\end{aligned}$$

Poles: $p_1 = -10$ and $p_2 = -2$. Zero: $z_1 = +10$. At this point we may already observe that the two poles are on the left–hand half–plane, so the system is stable; but the zero is on the right–hand half–plane, so the system is non–minimum phase (we'll pick up again on that point in question 5.)



4. (25%) Time domain–step response.

Answer: We can see from the transfer function that this is a second order system with two real & negative poles; therefore, we recognize an overdamped system. To obtain the step response, we multiply the transfer function by $1/s$, *i.e.* the Laplace transform of the unit step function, and expand in partial fractions.

$$-\frac{s-10}{s(s+2)(s+10)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+10}.$$

The partial fraction expansion works out as follows:

$$\begin{aligned} A &= -\left.\frac{s-10}{(s+2)(s+10)}\right|_{s=0} = \frac{1}{2}; \\ B &= -\left.\frac{s-10}{s(s+10)}\right|_{s=-2} = -\frac{3}{4}; \\ C &= -\left.\frac{s-10}{s(s+2)}\right|_{s=-10} = +\frac{1}{4}. \end{aligned}$$

Therefore, the Laplace transform of the step response is:

$$\frac{1/2}{s} - \frac{3/4}{s+2} + \frac{1/4}{s+10}.$$

From this, we obtain immediately the step response in the time domain

$$f(t) = \left(\frac{1}{2} - \frac{3}{4}e^{-2t} + \frac{1}{4}e^{-10t}\right)u(t).$$

5. (15%) Sketch the time response.

Answer: Looking at the limit $t \to \infty$ in the time–domain step response (or, equivalently, using the final value theorem on the Laplace transform of the step response) we find that the steady–state value of the step response is 1/2. Since this is an overdamped system, and there is no clear dominant pole, the response will approach the steady–state value asymptotically but we cannot easily specify by hand the time constant.

We can, however, say more about the behavior of the step response at small times ($t = 0+$.) Since the system is non–minimum phase (*i.e.*, it has a zero on the right–hand half–plane), we should be worried that its step response might initially head in the opposite direction (negative) than the steady–state value (positive.) We verify this by computing the slope (derivative $\mathrm{d}\omega(t)/\mathrm{d}t$) of the step response at $t = 0+$, as asked by the problem statement. Using the initial value theorem,

$$\mathcal{L}\left[\frac{\mathrm{d}\omega}{\mathrm{d}t}\right] = sF(s).$$

$$\left.\frac{\mathrm{d}\omega}{\mathrm{d}t}\right|_{t=0+} = \lim_{s\to\infty} s(sF(s)) = \lim_{s\to\infty} -s\frac{s(s-10)}{s(s+2)(s+10)} = +1.$$

(Note: alternatively, we can get the same answer by direct differentiation of the time–domain step response that we derived in question 4.) Indeed, the system exhibits an "undershoot," *i.e.* the motor starts rotating in the opposite direction (negative) than its steady–state value (positive.)

We have plotted below the exact response of the system using MATLAB . In your answer sheet, you should have drawn something that looks *approximately* like this, indicating the undershoot and the steady–state value.