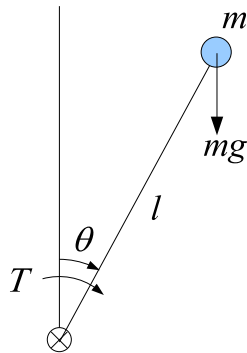


1. Inverted Pundulum

- a) *Answer:* From the following free body diagram, the equation of motion can be found by torque balance between the torque due to gravity, $mg \times l$; inertial torque $ml^2 \times \ddot{\theta}$; and the externally applied torque $T(t)$. We find



$$ml^2\ddot{\theta}(t) = mgl \sin \theta(t) + T(t). \\ \Rightarrow \ddot{\theta}(t) - \frac{g}{l} \sin \theta(t) = \frac{T(t)}{ml^2}.$$

Approximating $\sin \theta(t) \approx \theta(t)$, we find the linearized equation of motion as

$$\Rightarrow \ddot{\theta}(t) - \frac{g}{l}\theta(t) = \frac{T(t)}{ml^2}.$$

- b) *Answer:* Laplace transforming, we obtain

$$s^2\Theta(s) - \frac{g}{l}\Theta(s) = \frac{T(s)}{ml^2} \Rightarrow \frac{\Theta(s)}{T(s)} = \frac{1/ml^2}{s^2 - g/l}.$$

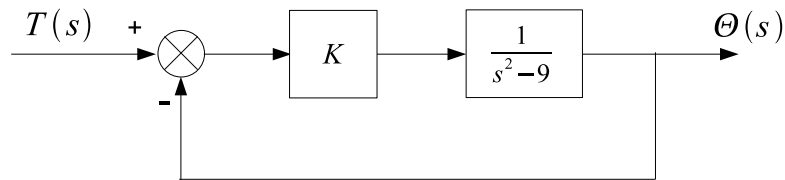
Using the given numerical values and $g = 9.81 (m/s^2)$, we find the transfer function as

$$\frac{\Theta(s)}{T(s)} = \frac{1}{s^2 - 9} = \frac{1}{(s + 3)(s - 3)}.$$

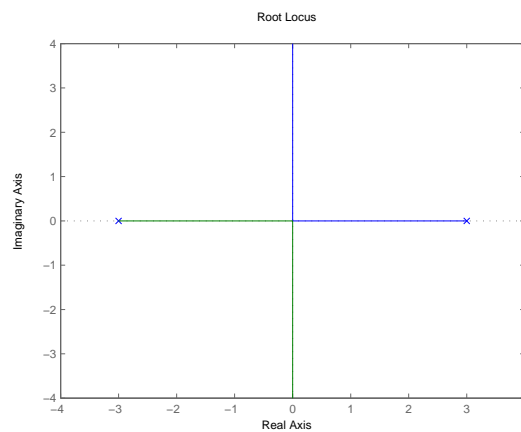
- c) *Answer:* The transfer function has a pole in the right-hand plane, thus the system is unstable.

Just like the example during lectures, stabilizing a stick so that it stands vertically without falling is a good example of an unstable system. The point is that if the stick were to stand perfectly vertical, it would not fall; but the slightest angular perturbation away from the vertical position (e.g. due to wind or slight motion of your hand, etc.) will grow rapidly causing the stick to fall from its vertical position. Because of its extreme sensitivity to perturbation the inverted pendulum is the archetypal unstable system.

- d) *Answer:* The block diagram with P controller.

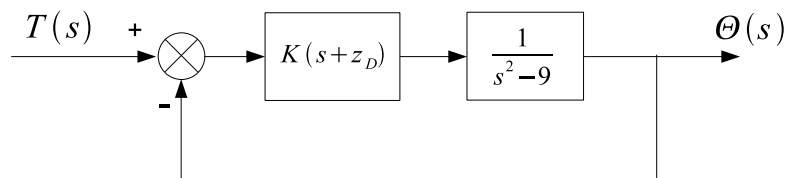


- e) *Answer:* The root locus of the system is as follows:

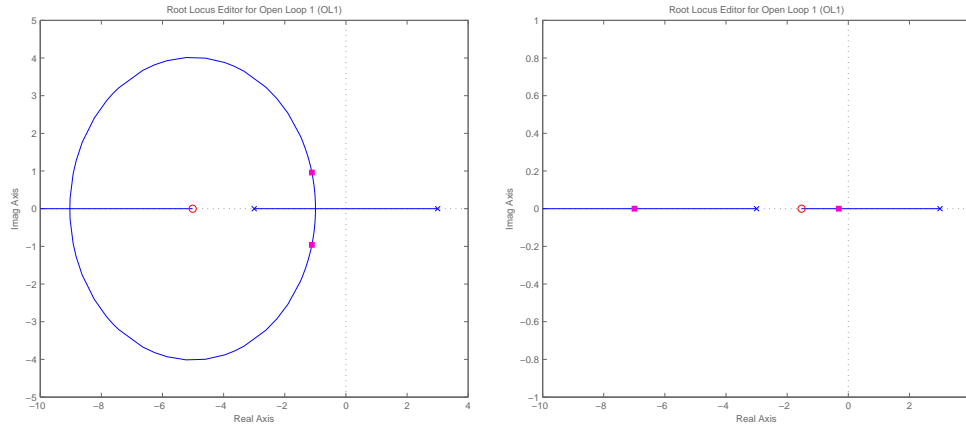


As the gain K increases, the closed-loop poles start moving from $s = \pm 3$ along the real axis. After they meet at $s = 0$ and move along the imaginary axis. When the poles are on the real axis, the system is unstable. When the poles are on the imaginary axis, the system is marginally stable. Thus this system cannot be stabilized with a proportional (P) controller.

- f) *Answer:* The block diagram with PD controller.



To move the closed-loop poles into the left-hand plane, you want to add a zero in left-hand plane. There are two possibilities:



The zero can be farther than $s = -3$ or in between $s = 0$ and $s = -3$. Either one can stabilize the system with a big gain. The second configuration (the zero in between the two poles) will generally need a much bigger gain to stabilize. So the first one (zero to the left of the stable pole) is more practical choice.

- g) *Answer:* The additional information is velocity. In the PD controller, “D” stands for derivative action. The error signal in the feedback loop (reference - output) contains velocity as well as displacement. Therefore the controller sees how big the error is *and also* how fast the error changes (the *rate* of error growth or decay.)

To understand why this is useful, consider the following: suppose that the error is big, but decreasing due to previous control input, then you can afford to use smaller control input because you don’t want to generate huge overshoot. Conversely, if the error is increasing because of a disturbance, the PD controller will detect the fast increase as well as the rate that the error is growing and succeed in compensating the error much better than the proportional (P) controller that take into account the error alone.

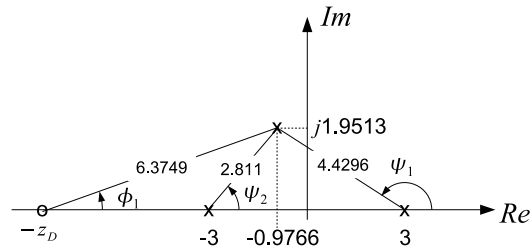
- h) *Answer:* From $T_p = 1.61$ (sec) and 20% OS specification,

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} = 0.4559, \quad \text{and}$$

$$T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = 1.61 \text{ (s)}.$$

Thus, the natural frequency $\omega_n = 2.1924$ (rad/s), the damped natural frequency $\omega_d = \omega_n \sqrt{1 - \zeta^2} = 1.9513$ (rad/s), and $\sigma = \zeta \omega_n = 0.9766$. The desired pair of poles are $s = -\sigma \pm j\omega_d = -0.9766 \pm j1.9513$.

- i) *Answer:* Using the idea of the root locus (closed-loop poles are the roots of the equation $1 + KG(s) = 0$, i.e. the roots of the denominator of the closed-loop transfer function), we consider following situation:

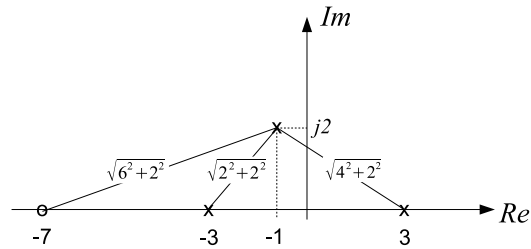


In the diagram, $-z_D$ is the zero that PD controller introduces. From the root locus drawing rules, $\psi_1 + \psi_2 - \phi_1 = 180^\circ$. Therefore, $\phi_1 = 17.8238^\circ$ and $z_D = 7.0455$. (These values can vary depending on the pole value. If you use $s = -1 + j2$, then you might have slightly different values.)

- j) *Answer:* Approximating $z_D \approx 7$ and the closed-loop pole location as $-1 \pm j2$, we can compute the gain geometrically as

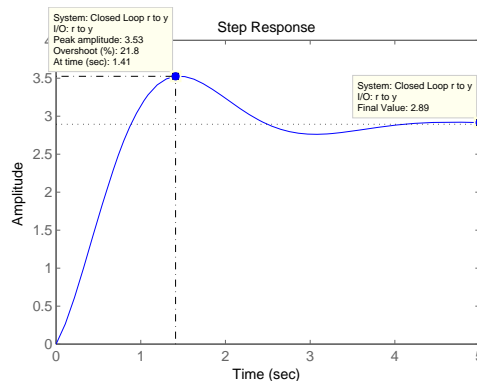
$$\frac{\sqrt{4^2 + 2^2} \sqrt{2^2 + 2^2}}{\sqrt{6^2 + 2^2}} = 2.$$

(see the sketch below.)



MATLAB gives the more accurate value 1.9519.

- k) *Answer:* From MATLAB's `sisotool`, the step response is as follows:



Note that $T_p = 1.41$ (s) and %OS is 21.8%. This is because the damping ratio ζ and natural frequency ω_n are computed by approximation of the actual

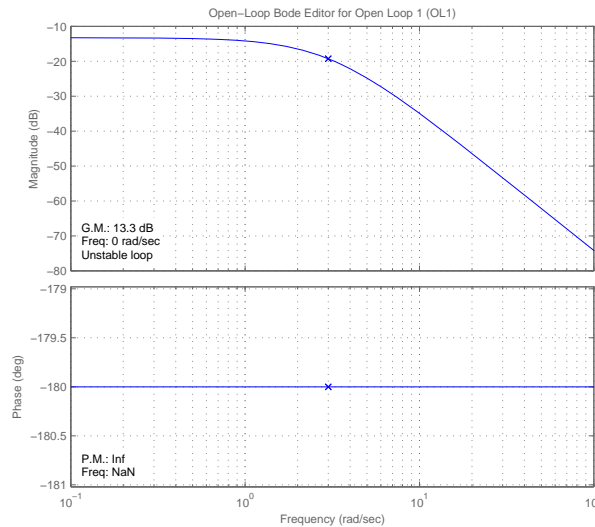
system as a 2nd order system whose poles are equal to the dominant poles of the actual system. The actual system has an additional zero due to the PD controller; therefore, we expect the exact response to be slightly different: derivative action will cause the peak time to become slightly shorter (faster) than the approximated system, and the overshoot to become slightly larger than the approximated system.

l) *Answer:* The steady state value θ_{ss} can be found as follows:

$$\theta(\infty) = \lim_{s \rightarrow 0} s \left(\frac{1}{s} \right) \frac{KG(s)}{1 + KG(s)} = \frac{7.0455K}{-9 + 7.0455K} = 2.8939,$$

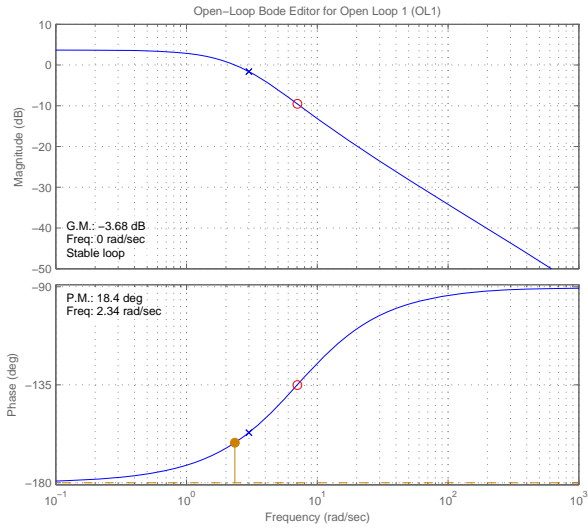
where $G(s) = \frac{s + 7.0455}{s^2 - 9}$ and $K = 1.9519$. Therefore, the steady state error $e(\infty) = 1 - y(\infty) = -1.8939$ or 189.39%. Notice that the exact step response found from MATLAB in question (k) agrees well with our estimate.

m) *Answer:* The open-loop bode plot of the uncompensated inverted pendulum is as follows:



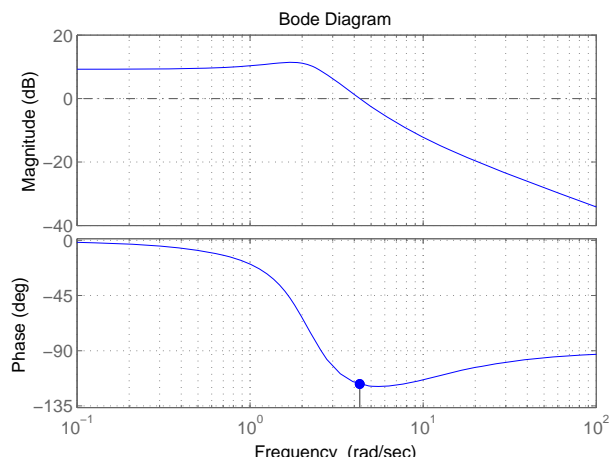
Phase is always 180°, the phase margin is zero. The gain margin is 13.3 dB in this case. Note that the gain margin depends on the gain K .

The open-loop bode plot of the PD-compensated inverted pendulum is as follows:



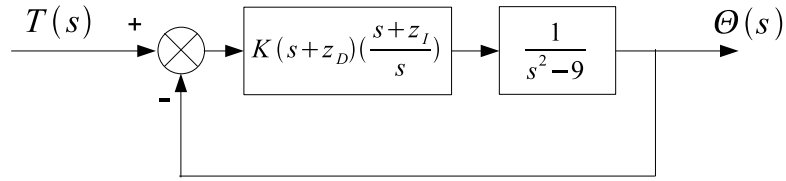
Phase at low frequencies is -180° , so the gain margin is -3.68 dB. The phase margin is 18.4° . This open-loop system is unstable, which explains why the gain margin can be negative while the phase margin is positive. Feedback produces a stable closed-loop system for appropriate choice of gain, as we found earlier. So the meaning of negative gain margin is that as the feedback gain in this system *increases*, the closed-loop system becomes stable.

The closed-loop bode plot of the PD-compensated inverted pendulum was not asked for in this question, but we provide it for clarity as follows:



The gain and phase margins are not meaningful for the closed-loop Bode plot; gain and phase margins are useful only to determine *closed loop stability from the open-loop Bode plots*.

n) *Answer:* We add a PI controller. The block diagram is as follows:



Note that the transfer function of the PI controller is $\frac{s+z_I}{s}$ not pure integrator. Since the question requires to use a safety factor of 10, $z_I = 0.3$.

The open-loop transfer function is now

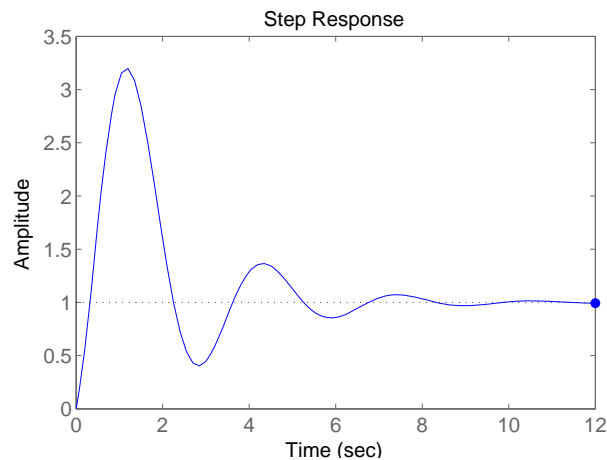
$$K(s+z_D)\frac{(s+z_I)}{s}\frac{1}{s^2-9}.$$

This is Type 1 system (includes one free integrator), so the steady state error to a step input is zero.

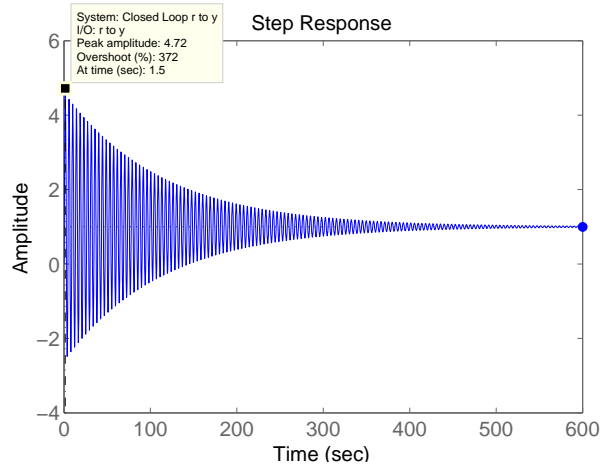
The steady state output is

$$y(\infty) = \lim_{s \rightarrow 0} s \left(\frac{1}{s} \right) \frac{K(s+z_D)\frac{(s+z_I)}{s}\frac{1}{s^2-9}}{1 + K(s+z_D)\frac{(s+z_I)}{s}\frac{1}{s^2-9}} = 1.$$

Therefore, the steady-state error $e(\infty) = 1 - y(\infty) = 0$.

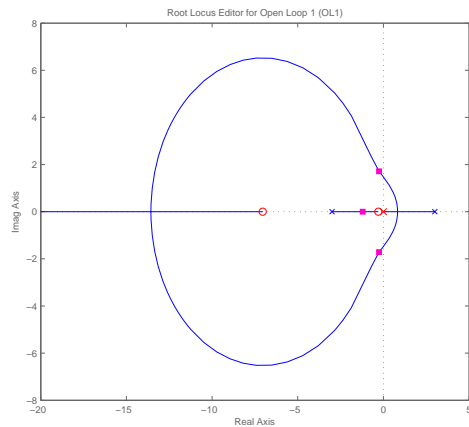


o) *Answer:* At the gain $K = 1.52$, the step response is as follows:

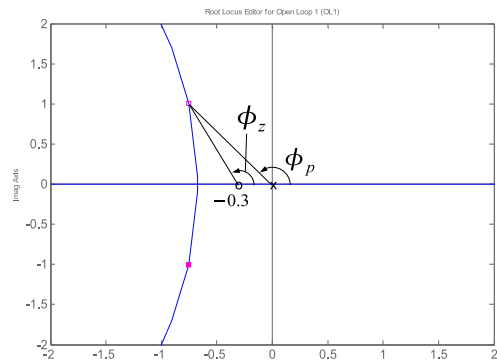


As you try to get the desired peak time as decreasing gain K , the overshoot becomes huge. Also when you decrease the gain more, then the system becomes unstable.

The reason for the excessive overshoot in the PID-compensated system is as follows: We note that the breakaway point of the PD-compensated system (before cascading the PI compensator) is at a very small negative value. This means that the root locus passes very close to the origin. Therefore, the usual PI controller assumption that the PI zero-pole pair does not have a significant total angular contribution to the (uncompensated) root locus is no longer true. Compare the schematics below.

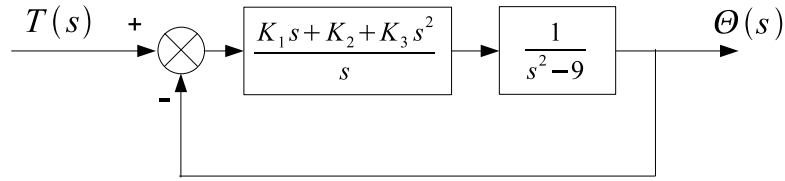


PID-compensated root locus



Zoom-in on the PD-compensated root locus (from question f) near the origin. Clearly, $\phi_z \neq \phi_p$.

- p) *Answer:* Now we control the individual gains of the P , I , and D components as follows:



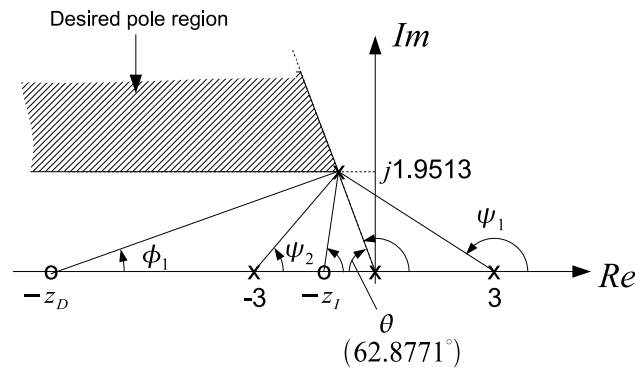
The transfer function of the PID controller is

$$\frac{K_1 s + K_2 + K_3 s^2}{s}.$$

From the new requirement, we obtain following relations:

- $\omega_d > 1.9513$.
- $0.2 > \exp\left\{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right\} \Rightarrow \zeta > 0.4559$ ($\cos\theta < 0.4559 \Rightarrow \theta < 62.8771^\circ$).

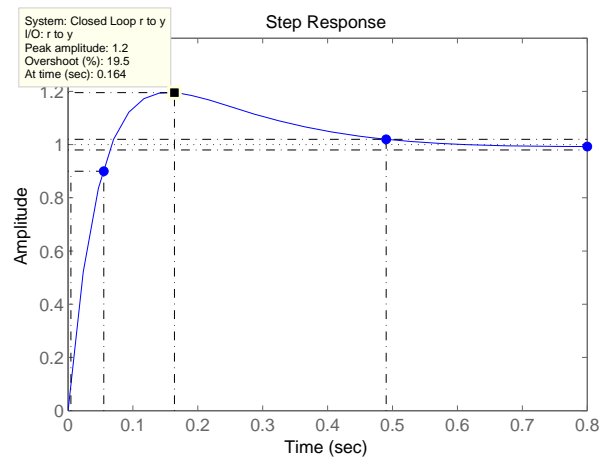
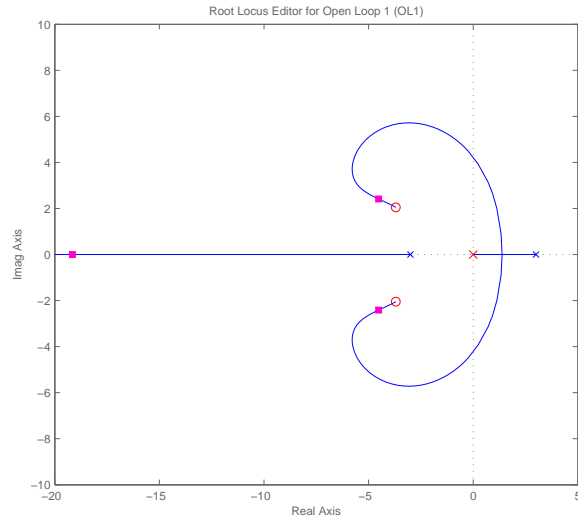
The desired pole location can be drawn as follows:



All you need to do in this problem is to find K_1 , K_2 , and K_3 that make the (dominant) closed-loop poles to be in the desired region. You can play with MATLAB and change K , z_D and z_I in the previous questions. Once you have a desired step response, then you can convert them to K_1 , K_2 , and K_3 .

By just changing these K , z_D , and z_I , you may not satisfy the desired specification.

Since we can change individual gains of K_1 , K_2 , and K_3 , we want to introduce another trick that you can play with. The transfer function of the compensator has the second order numerator, you can actually put a pair of complex zeros. For example, when you add a pair of zeros at $s = -3.7 \pm j2.04$ (and a pole at $s = 0$ of course), then you can satisfy the desired step response. We add the root locus and step response of this particular case.



In this particular case, $K_3 = 28.208$, $K_2 = 17.9$, and $K_1 = 7.4$.

2. Review questions

- a) *Answer:* $K_m = K_v$. This is because energy must be conserved when it is converted from the electrical to the mechanical domain by the DC Motor.
- b) *Answer:* Both spring and capacitor are energy storage elements (the spring stores elastic energy, the capacitor stores charge, *i.e.* electrostatic energy.)
- c) *Answer:* The damping ratio ζ specifies the 2nd-order system behavior. If $\zeta > 1$, the system is overdamped and there is no overshoot. If $\zeta = 1$, then system is critically damped. If $\zeta < 1$, then the system is underdamped and the step response is oscillation with exponential decay envelope. If $\zeta = 0$, then the system is undamped. The step response is pure oscillation without any decay.
- d) *Answer:* i) Yes ii) No.

Let's examine the transfer function of a single zero, $G(s) = (s \pm z)$. After substituting $s = j\omega$, the magnitude of $G(j\omega)$ is

$$|G(j\omega)| = \sqrt{z^2 + \omega^2},$$

while the phase of $G(j\omega)$ is

$$\angle G(j\omega) = \tan^{-1} \left(\frac{\omega}{\pm z} \right) = \pm \tan^{-1} \left(\frac{\omega}{z} \right).$$

Therefore, the Bode magnitude plot of a single zero always indicates 20 dB/decade while the phase plot indicates $\pm 90^\circ$ depending on where the zero is. If the zero is on the left-hand plane, then the phase Bode plot will change by $+90^\circ$ over a 2 decade span. If the zero is on the right-hand plane, then the phase Bode plot will change by -90° over a 2 decade span.

- e) *Answer:* The overshoot only depends on the damping ratio ζ . Therefore, our goal is to find ζ from either the root locus or the Bode plot.

In the root locus, the angle θ (subtended by the pole to the negative real axis) is related to the damping ratio ζ as $\cos \theta = \zeta$. Therefore, measuring this angle determines the damping ratio.

In the phase Bode plot, the phase margin is related to the damping ratio by

$$\Phi_M = \tan^{-1} \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^2}}}.$$

Therefore, measuring the phase margin also determines the damping ratio.