MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Quantum Computation

Problem 1. For any unit vector $\mathbf{j} = (j_x, j_y, j_z)$ we can define the following operator

$$\sigma_{\mathbf{j}} = j_x \sigma_X + j_y \sigma_Y + j_z \sigma_Z$$

which corresponds to a π-radian rotation about **j**-axis.

(a) Show that $\sigma_{\mathbf{j}}^2 = I$.

(b) $\sigma_{\mathbf{j}}$ has two eigenvalues: +1 and –1.

(c) Find the eigenvectors $|+\rangle_{\mathbf{j}}$ and $|-\rangle_{\mathbf{j}}$ respectively corresponding to eigenvalues +1 and –1.

Solution:

(a) Using the anti-commutator relation for Pauli matrices,

$$\{\sigma_i, \sigma_j\} \equiv \sigma_i \sigma_j + \sigma_j \sigma_i = 0 \text{ for } i \neq j \in \{X, Y, Z\}$$

and the fact that $\sigma_X^2 = \sigma_Y^2 = \sigma_Z^2 = I$ we have

$$\begin{aligned} \sigma_{\mathbf{j}}^2 &= j_x^2 \sigma_X^2 + j_y^2 \sigma_Y^2 + j_z^2 \sigma_Z^2 \\ &\quad + j_x j_y \{\sigma_X, \sigma_Y\} + j_z j_y \{\sigma_Y, \sigma_Z\} + j_x j_z \{\sigma_Z, \sigma_X\} \\ &= (j_x^2 + j_y^2 + j_z^2) I \\ &= I \,. \end{aligned}$$

(b) If $|\beta\rangle$ is an eigenstate of $\sigma_{\mathbf{j}}$ with eigenvalue β, then we have

$$\begin{aligned} \sigma_{\mathbf{j}} |\beta\rangle &= \beta |\beta\rangle \\ \Rightarrow \sigma_{\mathbf{j}}^2 |\beta\rangle &= \beta \sigma_{\mathbf{j}} |\beta\rangle \\ \Rightarrow I |\beta\rangle &= \beta^2 |\beta\rangle \\ \Rightarrow \beta^2 &= 1 \\ \Rightarrow \beta &= \pm 1 \end{aligned}$$

(c) In the basis $\{|0\rangle, |1\rangle\}$,

$$\sigma_{\mathbf{j}} = j_x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + j_y \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + j_z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} j_z & j_x - ij_y \\ j_x + ij_y & -j_z \end{bmatrix}$$

Therefore, assuming $|+\rangle_{\mathbf{j}} = \begin{bmatrix} a \\ b \end{bmatrix}$, $a \in \mathbb{R}^+$, and $|a|^2 + |b|^2 = 1$, we have

$$\sigma_{\mathbf{j}} |+\rangle_{\mathbf{j}} = |+\rangle_{\mathbf{j}}$$

$$\Rightarrow \quad aj_z + b(j_x - ij_y) = a$$

$$\Rightarrow \quad b = a\frac{1 - j_z}{j_x - ij_y}$$

$$\Rightarrow \quad \left(1 + \frac{(1 - j_z)^2}{j_x^2 + j_y^2}\right) a^2 = 1$$

$$\Rightarrow \quad \frac{1 - j_z^2 + 1 - 2j_z + j_z^2}{1 - j_z^2} a^2 = 1$$

$$\Rightarrow \quad a = \sqrt{(1 + j_z)/2}$$

$$\Rightarrow \quad |+\rangle_{\mathbf{j}} = \sqrt{\frac{(1 + j_z)}{2}} \begin{bmatrix} 1 \\ (1 - j_z)/(j_x - ij_y) \end{bmatrix}$$

Similarly, one can obtain

$$|-\rangle_{\mathbf{j}} = \sqrt{\frac{(1 + j_z)}{2}} \begin{bmatrix} 1 \\ -(1 + j_z)/(j_x - ij_y) \end{bmatrix}$$

You can verify that ${}_{\mathbf{j}}\langle + | - \rangle_{\mathbf{j}}=0$. You can also verify that $|+\rangle_{\mathbf{j}}$ and $|-\rangle_{\mathbf{j}}$ are respectively the unit vectors in the positive and negative directions of the **j**-axis on the Bloch sphere. This fact was predictable because these are the only two vectors that keep their orientations unchanged under the rotation about **j**-axis.

Problem 2. Find an approximation to

$$e^{i\theta\sigma_Y/2} e^{i\theta\sigma_X/2} e^{-i\theta\sigma_Y/2} e^{-i\theta\sigma_X/2}$$

up to the second order of θ for $\theta \ll 1$.

Solution:

Defining

$$f(\theta) = e^{i\theta\sigma_Y/2} e^{i\theta\sigma_X/2}$$
$$\simeq (I + i\theta\sigma_Y/2 - \theta^2\sigma_Y^2/8)(I + i\theta\sigma_X/2 - \theta^2\sigma_X^2/8)$$

$$
\begin{aligned}
&= (I + i\theta\sigma_Y/2 - \theta^2 I/8)(I + i\theta\sigma_X/2 - \theta^2 I/8) \\
&\simeq I - \theta^2 I/4 + i\theta(\sigma_X + \sigma_Y)/2 - \theta^2\sigma_Y\sigma_X/4 \\
&= I - \theta^2 I/4 + i\theta(\sigma_X + \sigma_Y)/2 + i\theta^2\sigma_Z/4
\end{aligned}
$$

we have

$$
\begin{aligned}
e^{i\theta\sigma_Y/2}e^{i\theta\sigma_X/2}e^{-i\theta\sigma_Y/2}e^{-i\theta\sigma_X/2} &= f(\theta)f(-\theta) \\
&\simeq I - \theta^2 I/2 + i\theta^2\sigma_Z/2 + \theta^2(\sigma_X + \sigma_Y)^2/4
\end{aligned}
$$

But using Problem 1.a for $j_x = j_y = 1/\sqrt{2}$, and $j_z = 0$, we have

$$(\sigma_X + \sigma_Y)^2/2 = I\,.$$

Therefore,

$$e^{i\theta\sigma_Y/2}e^{i\theta\sigma_X/2}e^{-i\theta\sigma_Y/2}e^{-i\theta\sigma_X/2} \simeq I + i\theta^2\sigma_Z/2$$

where all $\simeq$ signs stand for the approximation up to the second order of θ.

Problem 3. Rewrite $\frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B) \equiv |\psi\rangle$ in the following basis

$$\{|+\rangle_A \otimes |+\rangle_B, |+\rangle_A \otimes |-\rangle_B, |-\rangle_A \otimes |+\rangle_B, |-\rangle_A \otimes |-\rangle_B\}.$$

What are $\Pr(++)$, $\Pr(+-)$, $\Pr(-+)$, and $\Pr(--)$?

Solution:

We need to find the inner products of $|\psi\rangle$ and the basis vectors. Using the fact that

$$\langle +|0\rangle = \langle +|1\rangle = \langle -|0\rangle = -\langle -|1\rangle = 1/\sqrt{2}$$

we have

$$
\begin{aligned}
{}_B\langle +| \otimes {}_A\langle +|\psi\rangle &= \frac{1}{\sqrt{2}}[\,{}_A\langle +|0\rangle_A\ {}_B\langle +|0\rangle_B + {}_A\langle +|1\rangle_A\ {}_B\langle +|1\rangle_B\,] \\
&= 1/\sqrt{2} \\
{}_B\langle -| \otimes {}_A\langle +|\psi\rangle &= \frac{1}{\sqrt{2}}[\,{}_A\langle +|0\rangle_A\ {}_B\langle -|0\rangle_B + {}_A\langle +|1\rangle_A\ {}_B\langle -|1\rangle_B\,] \\
&= 0 \\
{}_B\langle +| \otimes {}_A\langle -|\psi\rangle &= \frac{1}{\sqrt{2}}[\,{}_A\langle -|0\rangle_A\ {}_B\langle +|0\rangle_B + {}_A\langle -|1\rangle_A\ {}_B\langle +|1\rangle_B\,] \\
&= 0 \\
{}_B\langle -| \otimes {}_A\langle -|\psi\rangle &= \frac{1}{\sqrt{2}}[\,{}_A\langle -|0\rangle_A\ {}_B\langle -|0\rangle_B + {}_A\langle -|1\rangle_A\ {}_B\langle -|1\rangle_B\,]
\end{aligned}
$$

$$= 1/\sqrt{2}$$

Therefore,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_A \otimes |+\rangle_B + |-\rangle_A \otimes |-\rangle_B).$$

Consequently,

$$\Pr(++) = \Pr(--) = 1/2$$

and

$$\Pr(+-) = \Pr(-+) = 0.$$

Problem 4. For any two qubits $|\psi\rangle$ and $|\phi\rangle$, find a unitary operator U with the following property:

$$U|\psi\rangle_A \otimes |\phi\rangle_B = |\phi\rangle_A \otimes |\psi\rangle_B.$$

Write down U in the basis $\{|0\rangle_A \otimes |0\rangle_B, |0\rangle_A \otimes |1\rangle_B, |1\rangle_A \otimes |0\rangle_B, |1\rangle_A \otimes |1\rangle_B\}$.

Solution:

To find the matrix representation of an operator *U*, it suffices to find its action on the basis vectors. In general, if the set of basis vectors is $\{|\psi_i\rangle, i \in I\}$ for an index set *I*, we have $U_{ij} = \langle\psi_i|U|\psi_j\rangle$, where U_{ij} is the element on the *i*th row and the *j*th column of the matrix representation of *U*. Therefore, using the following relations obtained from the main property of *U*,

$$\begin{aligned} U|0\rangle_A \otimes |0\rangle_B &= |0\rangle_A \otimes |0\rangle_B \\ U|0\rangle_A \otimes |1\rangle_B &= |1\rangle_A \otimes |0\rangle_B \\ U|1\rangle_A \otimes |0\rangle_B &= |0\rangle_A \otimes |1\rangle_B \\ U|1\rangle_A \otimes |1\rangle_B &= |1\rangle_A \otimes |1\rangle_B \end{aligned}$$

one can obtain

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For instance,

$$\begin{aligned} U_{23} &= {}_B\langle 1| \otimes {}_A\langle 0|U|1\rangle_A \otimes |0\rangle_B \\ &= {}_B\langle 1| \otimes {}_A\langle 0|0\rangle_A \otimes |1\rangle_B \\ &= 1 \end{aligned}$$

but,

$$
\begin{aligned}
U_{34} &= {}_B\langle 0| \otimes {}_A\langle 1|U|1\rangle_A \otimes |1\rangle_B \\
&= {}_B\langle 0| \otimes {}_A\langle 1|1\rangle_A \otimes |1\rangle_B \\
&= {}_A\langle 1|1\rangle_A \times {}_B\langle 0|1\rangle_B \\
&= 0\,.
\end{aligned}
$$

Problem 5. Write out

$$|\psi\rangle_{ABC} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_C + |1\rangle_A \otimes |0\rangle_C) \otimes |+\rangle_B$$

in the following basis:

$$\{|000\rangle_{ABC}, |001\rangle_{ABC}, |010\rangle_{ABC}, |011\rangle_{ABC}, |100\rangle_{ABC}, |101\rangle_{ABC}, |110\rangle_{ABC}, |111\rangle_{ABC}\}.$$

Solution:

$$
\begin{aligned}
|\psi\rangle_{ABC} &= \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_C + |1\rangle_A \otimes |0\rangle_C) \otimes \frac{1}{\sqrt{2}}(|0\rangle_B + |1\rangle_B) \\
&= \frac{1}{2}(|001\rangle_{ABC} + |100\rangle_{ABC} + |011\rangle_{ABC} + |110\rangle_{ABC})
\end{aligned}
$$

Problem 6.
Solution:

$${}_A\langle +|(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B)/\sqrt{2} = |+\rangle_B/\sqrt{2}\,.$$