Usecalctitle

Some Useful Results from Calculus

Derivation of Gauss' Theorem

Let $f(x, y, z)$ be a differentiable scalar function of (x, y, z).

$$\vec{f}_1 \equiv \hat{\imath} f$$

By the divergence theorem,

$$\int_V \nabla \cdot \vec{f}_1 \, dv = \int_S \vec{f}_1 \cdot \vec{n} \, ds = \int_S n_x f \, ds$$

$$\nabla \cdot \vec{f}_1 = \frac{\partial f}{\partial x}$$

$$\int_V \hat{\imath} \, \nabla \cdot \vec{f}_1 \, dv = \int_V \hat{\imath} \, \frac{\partial f}{\partial x} \, dv = \int_S \hat{\imath} \, n_x f \, ds$$

Similarly,

$$\int_V \hat{\jmath} \, \nabla \cdot \vec{f}_2 \, dv = \int_V \hat{\jmath} \, \frac{\partial f}{\partial y} \, dv = \int_S \hat{\jmath} \, n_y f \, ds$$

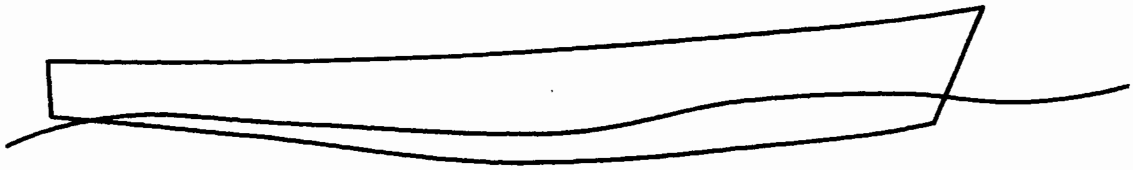
$$\int_V \hat{k} \, \nabla \cdot \vec{f}_3 \, dv = \int_V \hat{k} \, \frac{\partial f}{\partial z} \, dv = \int_S \hat{k} \, n_z f \, ds$$

Now, add the last three equations together,

$$\int_V \nabla f \, dv = \int_S \vec{n} f \, ds$$

gaussexp(1)

Example of Use of Gauss Theorem: Froude Krylov Surge Force on a Ship



$$P = \rho g A e^{-kz} \cos(kx - \omega t) \qquad\qquad \vec{F} = \int\int_S -p\vec{n}\, dS$$

$$F_x = \hat{i} \cdot \int\int_S -P\vec{n}\, ds = -\hat{i} \int\int\int_V \nabla P\, dV$$

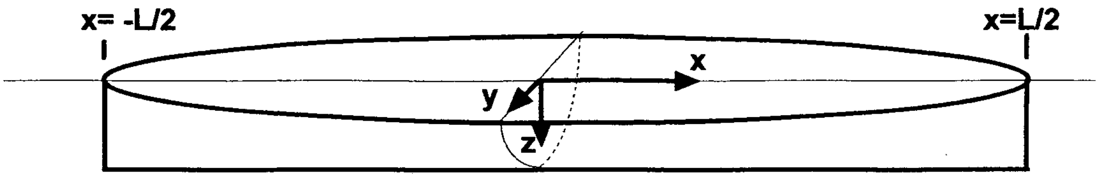
$$F_x = -\int\int\int_V \frac{\partial P}{\partial x}\, dV = \rho g A k \int\int\int_V e^{-kz} \sin(kx - \omega t) dV$$

$$\begin{aligned} F_x &\simeq \rho g A k \int\int\int_V (1 - kz) \sin(kx - \omega t) dV \\ &= \rho g A k \int_L \left[\int\int_{\text{section}} dy\, dz\right] \sin(kx - \omega t) dx \\ &\quad -\rho g A k^2 \int_L \left[\int\int_{\text{section}} z\, dy\, dz\right] \sin(kx - \omega t) dx \end{aligned}$$

$$F_x = \rho g A k \int_L S(x) \sin(kx - \omega t) dx - \rho g A k^2 \int_L z_{ca} S(x) \sin(kx - \omega t) dx$$

gauss exp(z)

Example with Given Ship Shape



$$y = \frac{2W}{L^2D^2}\left(\frac{L}{2}-x\right)^2 (D-z)^2$$

For this shape:

$$S(x) = \frac{2}{3}\frac{WD}{L^2}\left(\frac{L}{2}-x\right)^2 \qquad z_{ca} = \frac{D}{4}$$

Using these values:

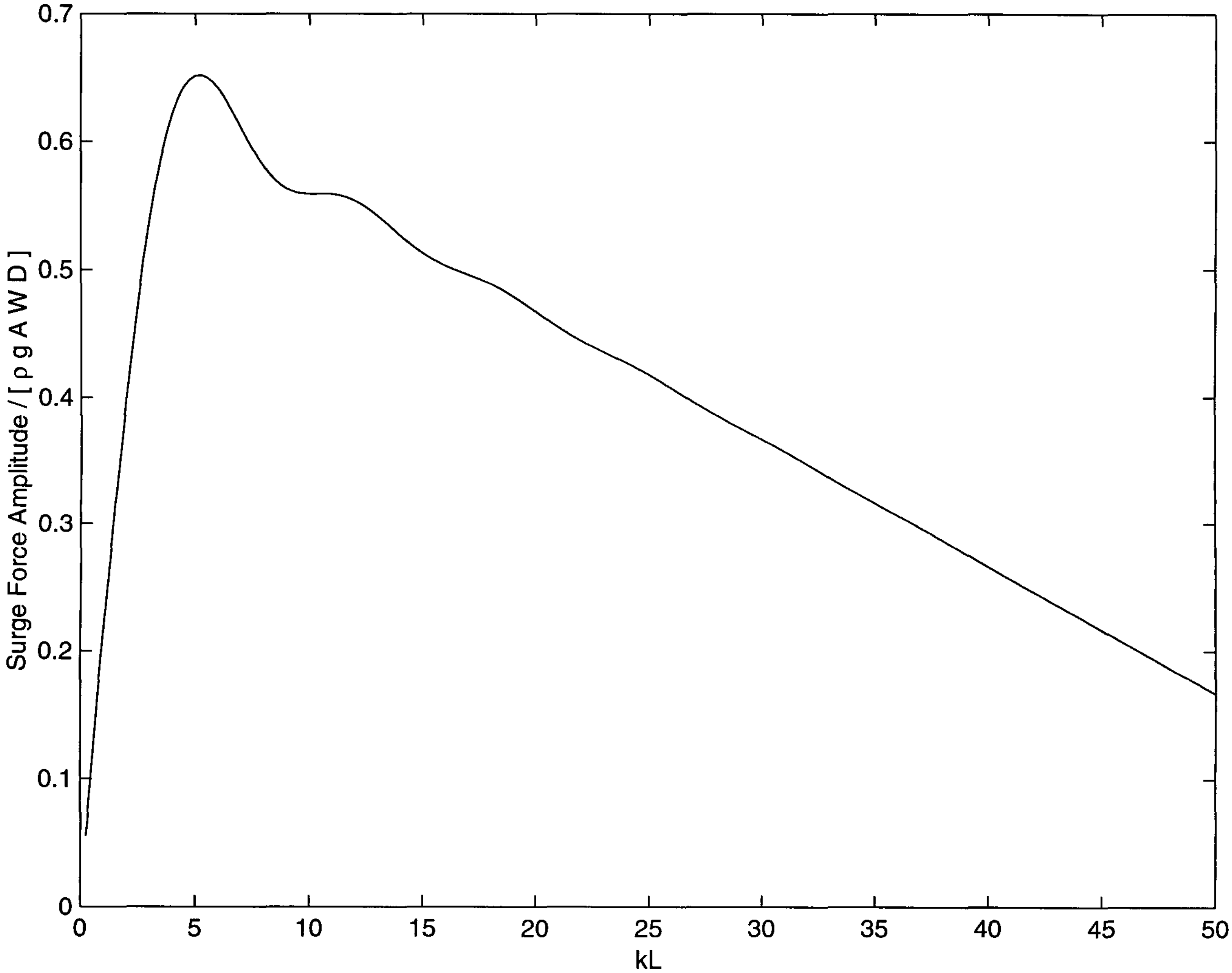
$$\begin{aligned} F_x &= \frac{2}{3}\rho g A k \frac{WD}{L^2}\left(1-\frac{kD}{4}\right)\left\{\cos\omega t\left[-\frac{2L}{k^2}\sin\frac{kL}{2}+\frac{L^2}{k}\cos\frac{kL}{2}\right]\right. \\ &\qquad \left. -\sin\omega t\left[\left(\frac{L^2}{k}-\frac{4}{k^3}\right)\sin\frac{kL}{2}+\frac{2L}{k^2}\cos\frac{kL}{2}\right]\right\} \end{aligned}$$

$$\begin{aligned} \frac{F_x}{\rho g A W D} &= \frac{2}{3}kL\left(1-\frac{kL}{4}\frac{D}{L}\right)\left\{\cos\omega t\left[-\frac{2}{(kL)^2}\sin\frac{kL}{2}+\frac{1}{kL}\cos\frac{kL}{2}\right]\right. \\ &\qquad \left. -\sin\omega t\left[\left(\frac{1}{kL}-\frac{4}{(kL)^3}\right)\sin\frac{kL}{2}+\frac{2}{(kL)^2}\cos\frac{kL}{2}\right]\right\} \end{aligned}$$

$$\begin{aligned} \left[\frac{F_x}{\rho g A W D}\right]_{\max} &= \frac{2}{3}kL\left(1-\frac{kL}{4}\frac{D}{L}\right)\left\{\left[-\frac{2}{(kL)^2}\sin\frac{kL}{2}+\frac{1}{kL}\cos\frac{kL}{2}\right]^2\right. \\ &\qquad \left. +\left[\left(\frac{1}{kL}-\frac{4}{(kL)^3}\right)\sin\frac{kL}{2}+\frac{2}{(kL)^2}\cos\frac{kL}{2}\right]^2\right\}^{1/2} \end{aligned}$$

```
% m-file script for gaussexp
tt = 2.0/3.0;
DoL = 0.06;
DoLf = DoL/4.;
m = 1:1:200;
kL = 0.25.*m;
kLi = 1.0 ./ kL;
sn = sin(kL ./ 2.0);
cs = cos(kL ./ 2.0);
fnd = tt .* kL .*(1.0 - kL .* DoLf) .* ((( -2 ./ (kL .^ 2)) .* sn + kLi .* ↙
cs) .^ 2 ...
  + ( ( kLi -4.0 .* (kLi .^3)) .* sn + (2.0 ./ (kL.^2)).* cs) .^2) .^ 0.5;
q = [kL;fnd];
fid = fopen('surge.dat','w');
fprintf(fid,'%8.3f, %9.4f\n',q);
fclose(fid);
plot(kL,fnd)
xlabel('kL')
ylabel('Surge Force Amplitude / [ \rho g A W D ]')
```

GAUSSEXP(?)
SURGE.dat



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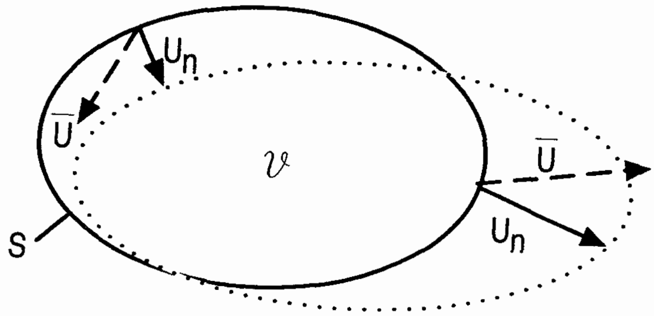
The Transport Theorem



Let $f(\mathbf{x}, t)$ be a differentiable scalar function of $\mathbf{x}$ and t.

Consider the integral,

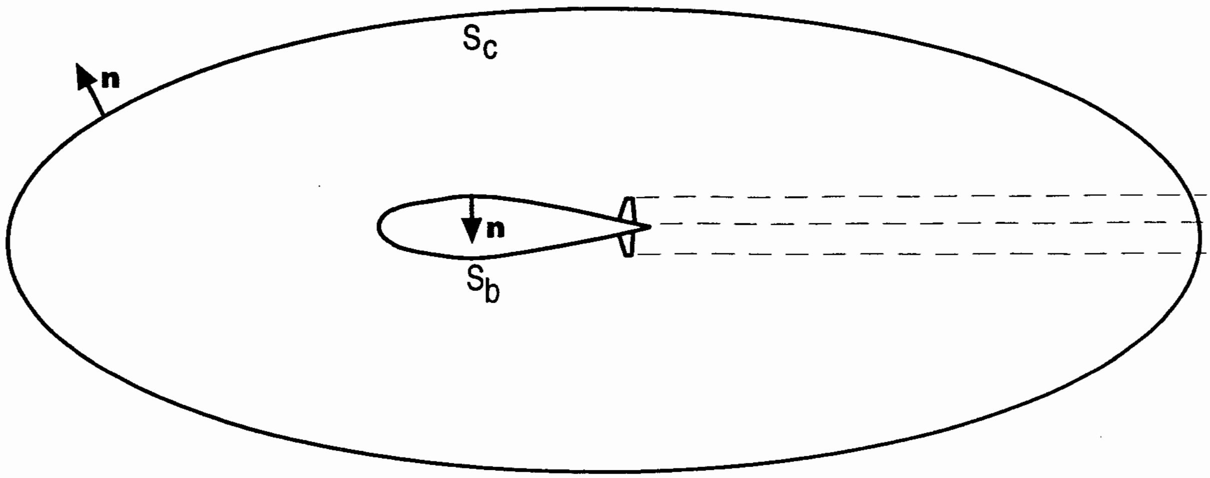
$$I(t) = \int\int\int_{\mathcal{V}(t)} f(\mathbf{x}, t) d\mathcal{V}$$

f is changing with time and $\mathcal{V}$ is changing with time. The normal component of the velocity of any point on the surface, S of $\mathcal{V}$ is called U_n.

$$\frac{dI}{dt} = \int\int\int_{\mathcal{V}} \frac{\partial f}{\partial t} d\mathcal{V} + \int\int_S f\, U_n dS = \int\int\int_{\mathcal{V}} \left\{ \frac{\partial f}{\partial t} + \nabla \cdot \left(f\, \vec{U} \right) \right\} d\mathcal{V}$$

Note that if $\vec{U}$ is the fluid velocity, the surface S is a material surface and the Transport Theorem is simply the integral form of the Substantial Derivative.

Pressure Forces and Moments on an Object



$$\mathbf{F} = \int\int_{S_b} p\,\mathbf{n}\,dS \qquad\qquad \mathbf{M} = \int\int_{S_b} p\,(\mathbf{r}\times\mathbf{n})\,dS$$

Now use the (unsteady) Bernoulli equation:

$$\mathbf{F} = -\rho\int\int_{S_b}\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi\right]\mathbf{n}\,dS$$

$$\mathbf{M} = -\rho\int\int_{S_b}\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi\right](\mathbf{r}\times\mathbf{n})\,dS$$

The following results from applying Gauss, Transport and divergence theorems and boundary conditions:

$$\mathbf{F} = -\rho\frac{d}{dt}\int\int_{S_b}\phi\,\mathbf{n}\,dS - \rho\int\int_{S_c}\left[\frac{\partial\phi}{\partial n}\nabla\phi - \mathbf{n}\frac{1}{2}\nabla\phi\cdot\nabla\phi\right]\,dS$$

$$\mathbf{M} = -\rho\frac{d}{dt}\int\int_{S_b}\phi\,(\mathbf{r}\times\mathbf{n})\,dS - \rho\int\int_{S_c}\mathbf{r}\times\left[\frac{\partial\phi}{\partial n}\nabla\phi - \mathbf{n}\frac{1}{2}\nabla\phi\cdot\nabla\phi\right]\,dS$$