Introduction to Numerical Analysis for Engineers

- **Roots of Non-linear Equations**
  - Heron's formula
  - Stop criteria
  - General method
    - Convergence
    - Examples
  - Newton-Raphson's Method
    Convergence Speed
    - Examples
  - Secant Method
    Convergence and efficiency
    - Examples
  - Multiple roots
  - Bisection

Roots of Nonlinear Equations

$$f(x) = 0$$

Example – Square root

$$x^2 - a = 0 \Rightarrow x = \sqrt{a}$$

Heron's Principle

$$x > 0$$

$$x^2 - a = 0 \Leftrightarrow x = \frac{a}{x}$$

Guess root

$$x_0 > \sqrt{a} \iff \frac{a}{x_0} < \sqrt{a}$$

$$x_0 < \sqrt{a} \iff \frac{a}{x_0} > \sqrt{a}$$

Mean is better guess

$$x_1 = (x_0 + \frac{a}{x_0})/2$$

Iteration Formula

$$x_k = (x_{k-1} + \frac{a}{x_{k-1}})/2$$

heron.m

```
a=2;
n=6;
g=2;
% Number of Digits
dig=5;
    sq(1)=g;
    for i=2:n
     sq(i)= 0.5*radd(sq(i-1),a/sq(i-1),dig);
    end
    '       i        value   '
    [ [1:n]' sq']
    hold off
    plot([0 n],[sqrt(a) sqrt(a)],'b')
    hold on
    plot(sq,'r')
    plot(a./sq,'r-.')
    plot((sq-sqrt(a))/sqrt(a),'g')
    grid on
```

i	value
1.0000	2.0000
2.0000	1.5000
3.0000	1.4167
4.0000	1.4143
5.0000	1.4143
6.0000	1.4143

Roots of Nonlinear Equations
General Method

Example: Cube root

Non-linear Equation

$$f(x) = 0$$

Goal: Converging series

$$x_0, x_1, \ldots x_n \to x^e, \quad n \to \infty$$

Rewrite Problem

$$f(x) = 0 \Leftrightarrow g(x^e) = x^e$$
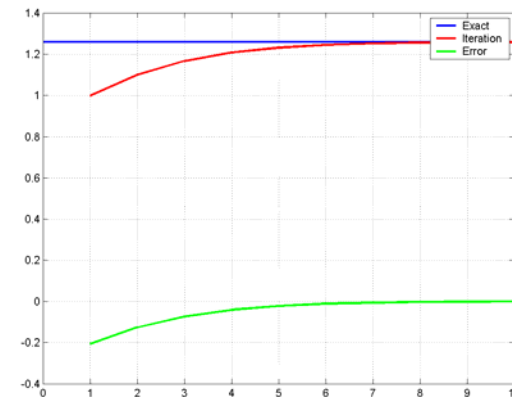
Example

$$g(x) = x + c \cdot f(x)$$

Iteration

$$x_n = g(x_{n-1})$$

```
% f(x) = x^3 - a = 0
% g(x) = x + C*(x^3 - a)        cube.m
a=2;
n=10;
g=1.0;
C=-0.1;
    sq(1)=g;
    for i=2:n
     sq(i)= sq(i-1) + C*(sq(i-1)^3 -a);
    end
    hold off
    plot([0 n],[a^(1./3.) a^(1/3.)],'b')
    hold on
    plot(sq,'r')
    plot( (sq-a^(1./3.))/(a^(1./3.)),'g')
    grid on
```

Roots of Nonlinear Equations
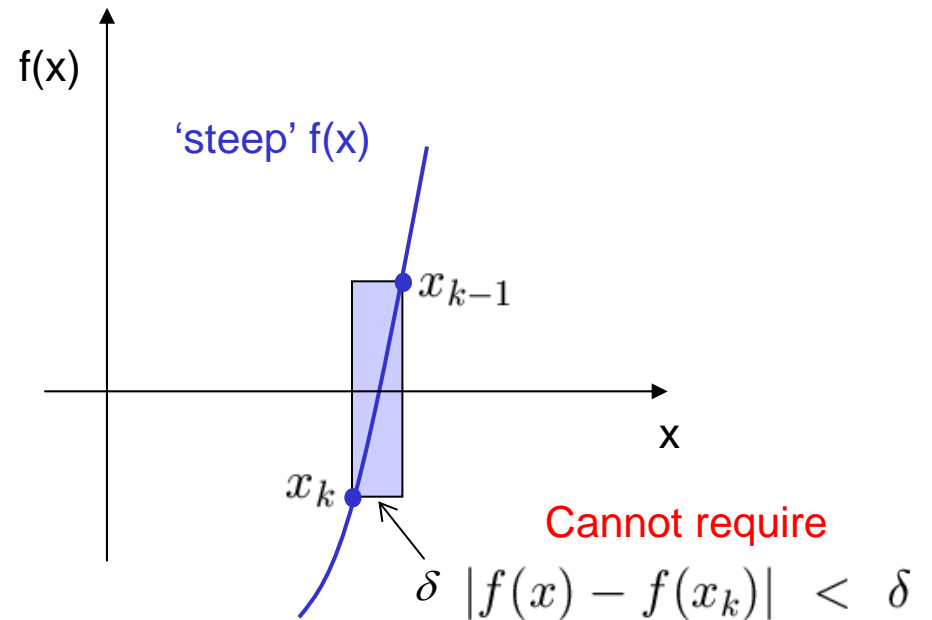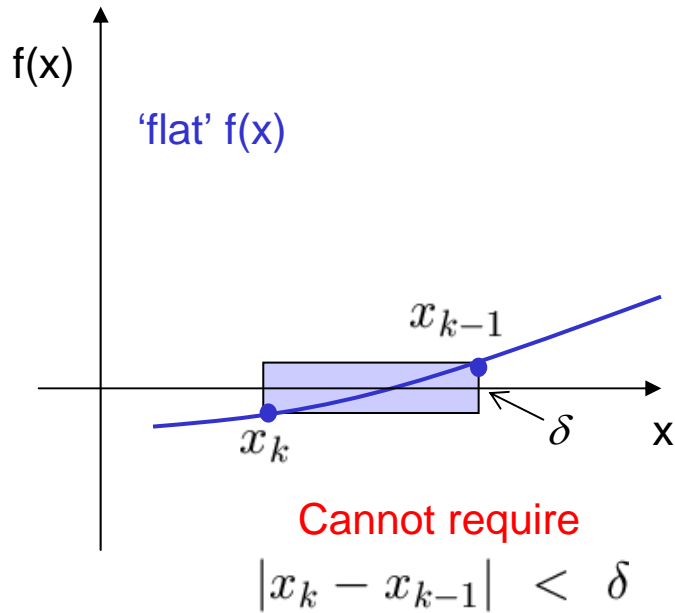## Stop-criteria

Unrealistic stop-criteria

$$x_{k+1} \neq x_k$$

Realistic stop-criteria

$$|x_k - x_{k-1}| < \delta \longleftarrow \text{Machine Accuracy}$$
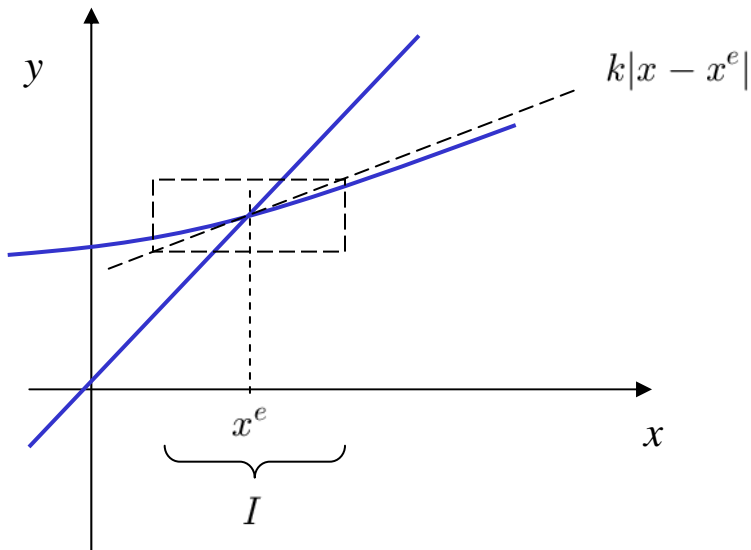
$$|f(x) - f(x_k)| < \delta$$

Use combination of the two criteria

f(x)

'flat' f(x)

x_{k-1}

x_k

δ

x

Cannot require

$$|x_k - x_{k-1}| < \delta$$

f(x)

'steep' f(x)

x_{k-1}

x_k

δ

x

Cannot require

$$|f(x) - f(x_k)| < \delta$$

Roots of Nonlinear Equations
General Method

Convergence

Define k such that if

$$x \in I$$

then

$$|g(x) - g(x^e)| = |g(x) - x^e| \leq k|x - x^e|$$

Convergence Criteria

$$x_{n-1} \in I \Rightarrow |x_n - x^e| = |g(x_{n-1}) - x^e| \leq k|x_{n-1} - x^e|$$

Apply successively

$$|x_n - x^e| \leq k^n|x_0 - x^e|$$

Convergence

$$x_0 \in I, \quad k < 1$$

Roots of Nonlinear Equations
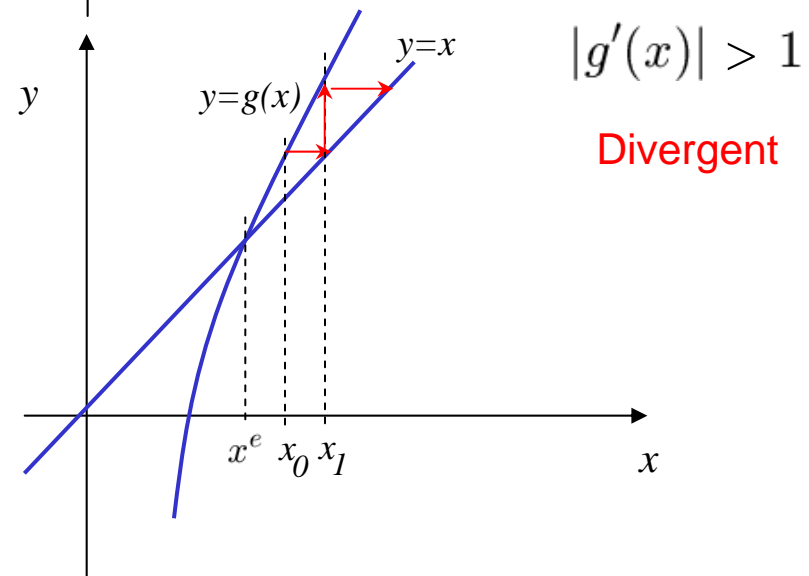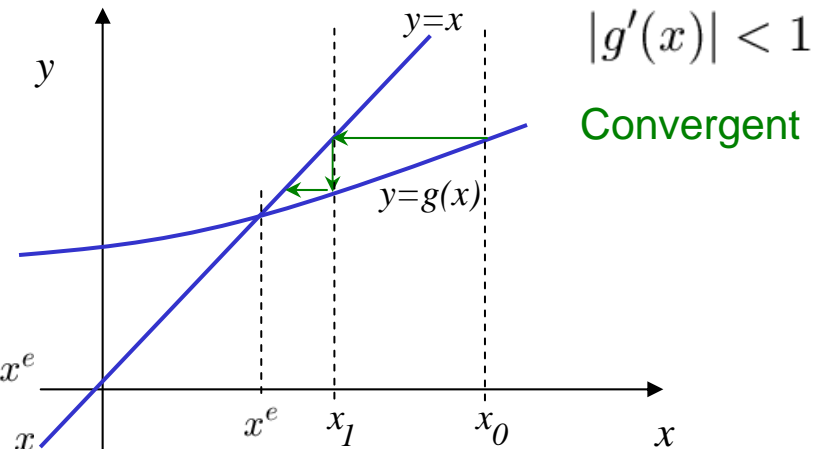## General Method

Convergence

$|g'(x)| < 1$

Convergent

Mean-value Theorem

$$\{\exists \xi \in [x, x^e] \mid g(x) - g(x^e) = g'(\xi)(x - x^e)\} \begin{cases} x < \xi < x^e \\ x^e < \xi < x \end{cases}$$

Convergence

$$|g'(x)|_{x \in I} \le k < 1 \Rightarrow |g(x) - x^e| \le k|x - x^e|$$

$|g'(x)| > 1$

Divergent

Roots of Nonlinear Equations
General Method

Example: Cube root

$$x^3 - 2 = 0 \ , \ \ x^e = 2^{1/3}$$

Rewrite

$$g(x) = x + C(x^3 - 2)$$

$$g'(x) = 3Cx^2 + 1$$

Convergence

$$|g'(x)| < 1 \ \Leftrightarrow \ -2 < 3Cx^2 < 0$$

$$\Leftrightarrow \boxed{-1/6 < C < 0}$$

$$C = -\frac{1}{6} \Rightarrow x_{n+1} = g(x_n) = x_n - \frac{1}{6}(x_n^3 - 2)$$

Converges more rapidly for small $|g'(x)|$

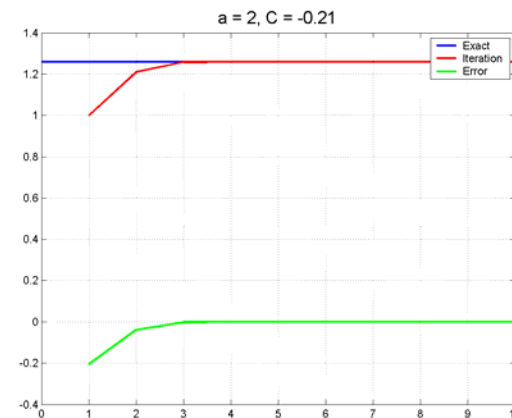$$g'(1.26) = 3C \cdot 1.26^2 + 1 = 0 \Leftrightarrow \boxed{C = -0.21}$$

cube.m

```
n=10;
g=1.0;
C=-0.21;
   sq(1)=g;
   for i=2:n
    sq(i)= sq(i-1) + C*(sq(i-1)^3 -a);
   end
   hold off
   f=plot([0 n],[a^(1./3.) a^(1/3.)],'b')
   set(f,'LineWidth',2);
   hold on
   f=plot(sq,'r')
   set(f,'LineWidth',2);
   f=plot( (sq-a^(1./3.))/(a^(1./3.)),'g')
   set(f,'LineWidth',2);
   legend('Exact','Iteration','Error');
   f=title(['a = ' num2str(a) ', C = ' num2str(C)])
   set(f,'FontSize',16);
   grid on
```


a = 2, C = -0.21

Roots of Nonlinear Equations
General Method

Converging, but how close?

$$
\begin{aligned}
|x_{n-1} - x^e| &\leq |x_{n-1} - x_n| + |x_n - x^e| \\
&= |x_{n-1} - x_n| + |g(x_{n-1}) - g(x^e)| \\
&= |x_{n-1} - x_n| + |g'(\xi)||x_{n-1} - x^e| \\
&\leq |x_{n-1} - x_n| + k|x_{n-1} - x^e| \\
&\Rightarrow
\end{aligned}
$$

$$
|x_{n-1} - x^e| \leq \frac{1}{1-k}|x_{n-1} - x_n|
$$

Absolute error

$$
|x_n - x^e| \leq k|x_{n-1} - x^e| \leq \frac{k}{1-k}|x_{n-1} - x_n|
$$

General Convergence Rule

$$
x_{n+1} = g(x_n)
$$

$$
|x_n - x^e| \leq \frac{k}{1-k}|x_{n-1} - x_n|
$$

$$
|g'(x)| < k < 1 \,, \quad x \in I
$$

Roots of Nonlinear Equations
Newton-Raphson Method

Non-linear Equation

$$f(x) = 0 \Leftrightarrow x = g(x)$$

Convergence Criteria

$$|g'(x_n)| < k < 1 \Rightarrow |x_n - x^e| \leq k|x_{n-1} - x^e|$$
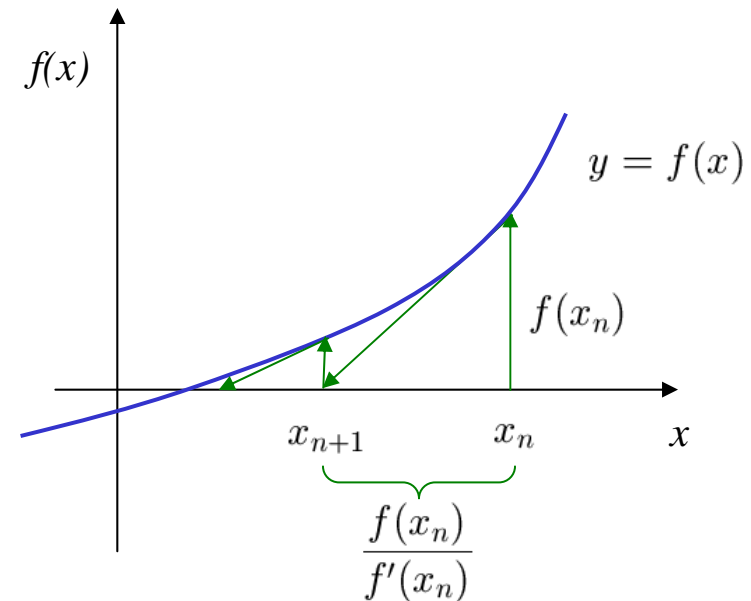
Fast Convergence

$$|g'(x^e)| = 0$$

$$g(x) = x + h(x)f(x), \quad h'(x) \neq 0$$

$$g'(x^e) = 1 + h(x^e)f'(x^e) + h'(x^e)f(x^e)$$
$$= 1 + h(x^e)f'(x^e)$$

$$g'(x^e) = 0 \Leftarrow h(x) = -\frac{1}{f'(x)}$$

Newton-Raphson Iteration

$$x_{n+1} = g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$$

$f(x)$

$y = f(x)$

$f(x_n)$

x_{n+1} x_n x

$$\frac{f(x_n)}{f'(x_n)}$$

Roots of Nonlinear Equations
Newton-Raphson Method

Example – Square Root

$$x = \sqrt{a} \Leftrightarrow f(x) = x^2 - a = 0$$

Newton-Raphson

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$
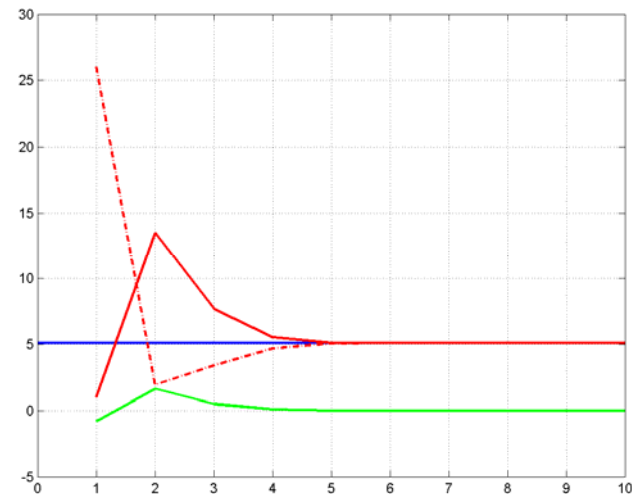
Same as Heron's formula

```
a=26;
n=10;                    sqr.m
g=1;
    sq(1)=g;
    for i=2:n
     sq(i)= 0.5*(sq(i-1) + a/sq(i-1));
    end
    hold off
    plot([0 n],[sqrt(a) sqrt(a)],'b')
    hold on
    plot(sq,'r')
    plot(a./sq,'r-.')
    plot((sq-sqrt(a))/sqrt(a),'g')
    grid on
```

Roots of Nonlinear Equations
Newton-Raphson Method

$$x = \frac{1}{a}$$

$$f(x) = ax - 1 = 0$$

$$f'(x) = a$$

Approximate Guess

$$\frac{|x - x^e|}{|x^e|} \ll 1$$

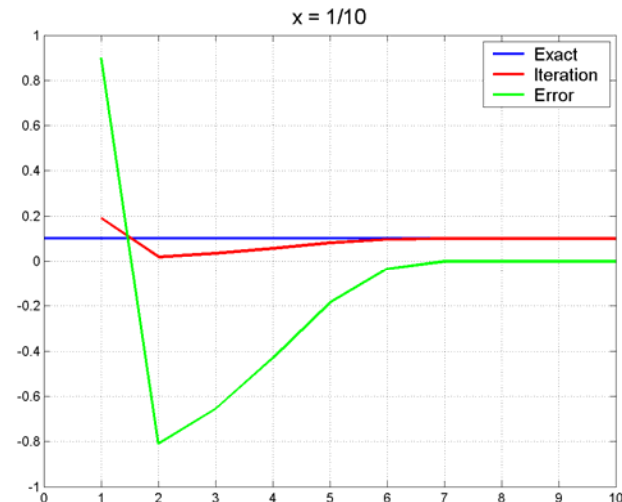$$\frac{f(x)}{f'(x)} = \frac{ax-1}{a} = x^e(ax-1) \simeq x(ax-1)$$

Newton-Raphson

$$x_{n+1} = x_n - x_n(ax_n - 1)$$

div.m

```
a=10;
n=10;
g=0.19;
sq(1)=g;
    for i=2:n
     sq(i)=sq(i-1) - sq(i-1)*(a*sq(i-1) -1) ;
    end
    hold off
    plot([0 n],[1/a 1/a],'b')
    hold on
    plot(sq,'r')
    plot((sq-1/a)*a,'g')
    grid on
    legend('Exact','Iteration','Error');
    title(['x = 1/' num2str(a)])
```

Roots of Nonlinear Equations
Newton-Raphson Method

$$\epsilon_n = x_n - x^e$$

Taylor Expansion

$$g(x_n) = g(x^e) + \epsilon_n g'(x^e) + \frac{1}{2}\epsilon_n^2 g''(x^e) \cdots$$

Second Order Expansion

$$g(x_n) - g(x^e) \simeq \frac{1}{2}\epsilon_n^2 g''(x^e)$$

$$\Rightarrow$$

$$\epsilon_{n+1} = x_{n+1} - x_e \simeq \frac{1}{2}\epsilon_n^2 g''(x^e)$$

Relative Error

$$\frac{\epsilon_{n+1}}{|x^e|} \simeq \frac{1}{2}|x^e|g''(x^e)\left(\frac{\epsilon_n}{|x^e|}\right)^2 = A(x^e)\left(\frac{\epsilon_n}{|x^e|}\right)^2 \qquad \boxed{\text{Quadratic Convergence}}$$

General Convergence Rate

$$\epsilon_{n+1} \simeq \epsilon_n^m A$$

Convergence Exponent

Roots of Nonlinear Equations
Secant Method

1. In Newton-Raphson we have to evaluate 2 functions $f_n(x)$, $f'_n(x)$

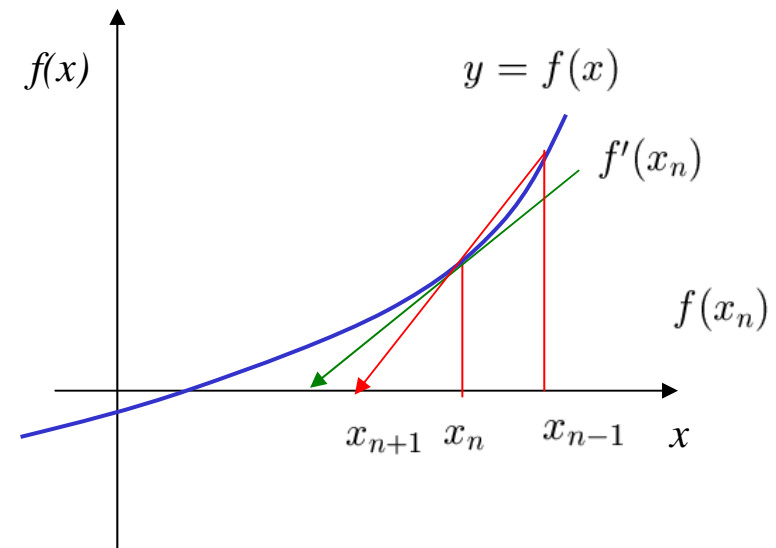2. $f_n(x)$ may not be given in closed, analytical form, i.e. it may be a result of a numerical algorithm

Approximate Derivative

$$f'(x_n) \simeq \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Secant Method Iteration

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$= \frac{f(x_n)x_{n-1} - f(x_{n-1})x_n}{f(x_n) - f(x_{n-1})}$$

Only 1 function call per iteration: $f_n(x)$

Roots of Nonlinear Equations
Secant Method

Convergence Speed

Absolute Error

$$\epsilon_n = x_n - x^e$$

$$\epsilon_{n+1} = x_{n+1} - x^e = \frac{f(x^e + \epsilon_n)(x^e + \epsilon_{n-1}) - f(x^e + \epsilon_{n-1})(x^e + \epsilon_n)}{f(x^e + \epsilon_n) - f(x^e + \epsilon_{n-1})} - x^e$$

Error Exponent

Taylor Series – 2nd order

$$\epsilon_n = A(x^e)\epsilon_{n-1}^m \Rightarrow \epsilon_{n-1} = \left(\frac{1}{A}\epsilon_n\right)^{1/m} = B(x^e)\epsilon_n^{1/m}$$

$$\epsilon_{n+1} \simeq \frac{1}{2}\epsilon_{n-1}\epsilon_n \frac{f''(x^e)}{f'(x^e)}$$

$$\epsilon_{n+1} = C(x^e)\epsilon_n\epsilon_{n-1} = D(x^e)\epsilon_n\epsilon_n^{1/m} = D(x^e)\epsilon_n^{1+1/m}$$

Relative Error

$$\frac{\epsilon_{n+1}}{|x^e|} \simeq \frac{\epsilon_{n-1}}{|x_e|}\frac{\epsilon_n}{|x_e|}\frac{f''(x^e)}{2f'(x^e)}x^e$$

$$1 + \frac{1}{m} = m \Leftrightarrow m = \frac{1}{2}(1 + \sqrt{5}) \simeq 1.62$$

Error improvement for each function call

Secant Method $\qquad \epsilon_{n+1}^* \simeq \epsilon_n^{1.62}$

Newton-Raphson $\quad \epsilon_{n+1}^* = \epsilon_n^{\sqrt{2}} \simeq \epsilon_n^{1.22}$

Exponents called Efficiency Index

Roots of Nonlinear Equations
Multiple Roots

p-order Root

$$f(x) = (x - x^e)^p f_1(x) \ , \ f_1(x^e) \neq 0$$

Newton-Raphson

$$x_{n+1} = g(x_n) = x_n - \frac{(x_n - x^e)^p f_1(x_n)}{p(x_n - x^e)^{p-1} f_1(x_n) + (x_n - x^e)^p f'(x_n)}$$
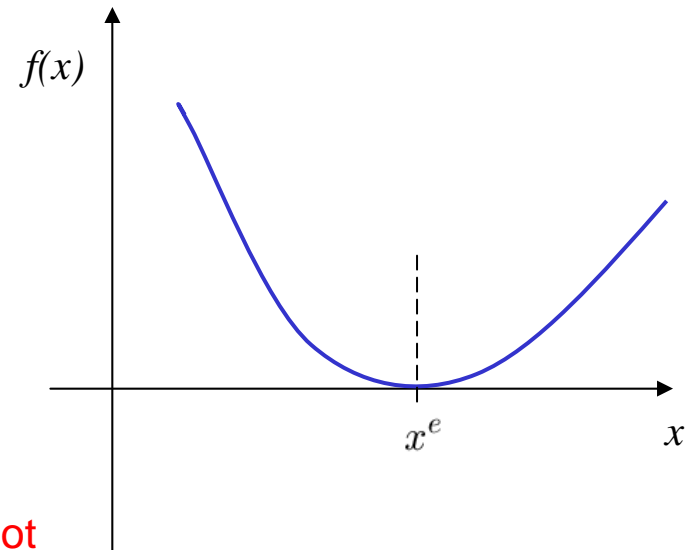
=>

$$x_{n+1} = x_n - \frac{(x_n - x^e) f_1(x_n)}{p f_1(x_n) + (x_n - x^e) f'(x_n)}$$

$f(x)$

Convergence

$$|x_{n+1} - x^e| \leq k|x_n - x^e| \simeq |g'(x^e)| \, |x_n - x^e|$$
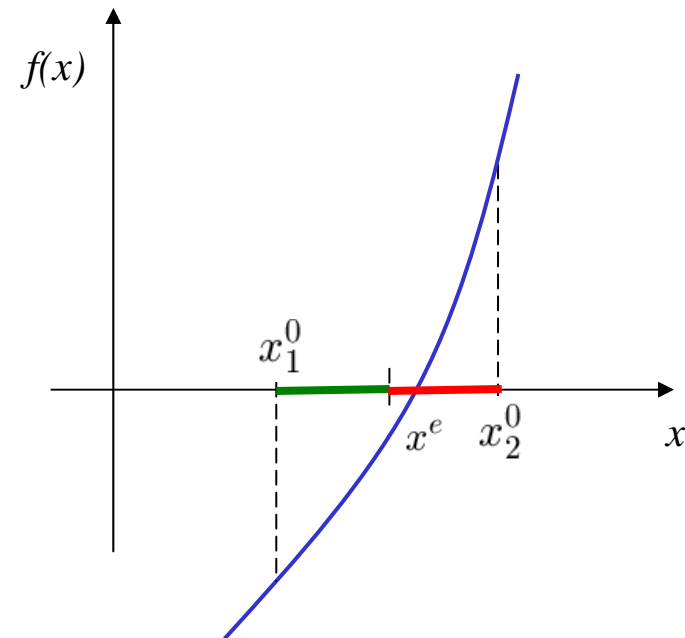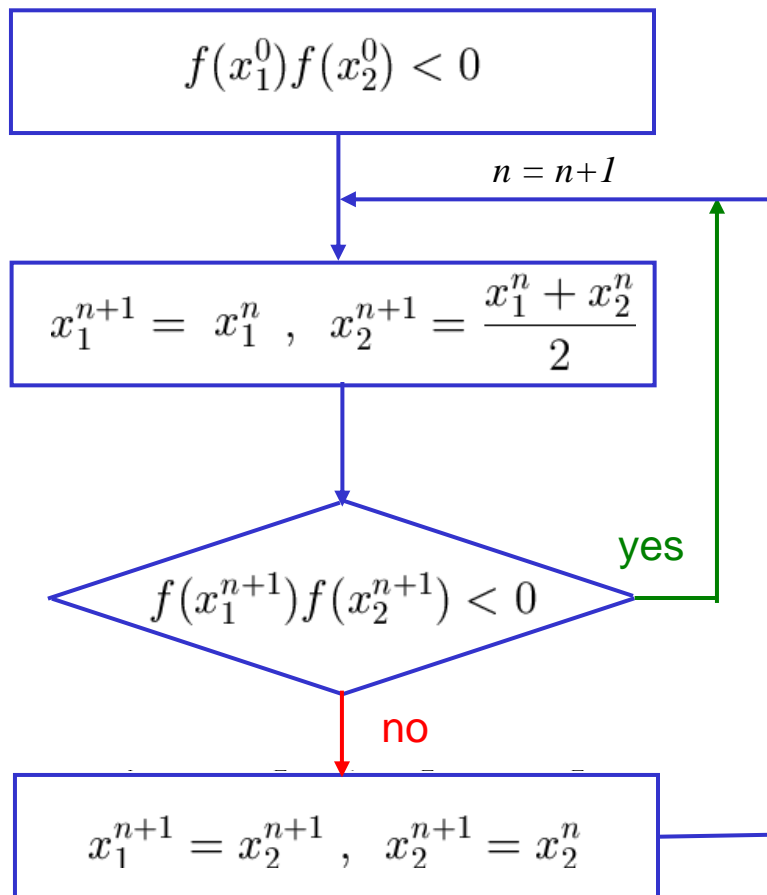
$$g'(x^e) = 1 - \frac{1}{p}$$

Slower convergence the higher the order of the root

x^e　　　x

Roots of Nonlinear Equations
Bisection

Algorithm

$$f(x_1^0)f(x_2^0) < 0$$

$n = n+1$

$$x_1^{n+1} = x_1^n \ , \quad x_2^{n+1} = \frac{x_1^n + x_2^n}{2}$$

$$f(x_1^{n+1})f(x_2^{n+1}) < 0$$

yes

no

$$x_1^{n+1} = x_2^{n+1} \ , \quad x_2^{n+1} = x_2^n$$

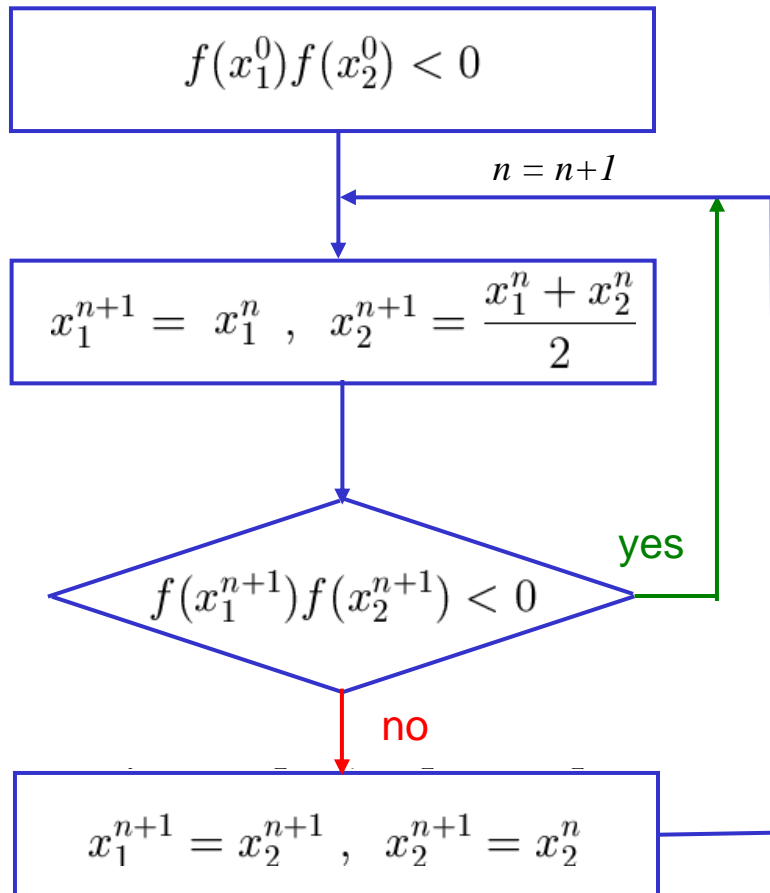$f(x)$

x_1^0

$x^e \quad x_2^0$

x

Less efficient than Newton-Raphson and Secant methods, but often used to isolate interval with root and obtain approximate value. Then followed by N-R or Secant method for accurate root.

Roots of Nonlinear Equations
Bisection

Algorithm

$$f(x_1^0)f(x_2^0) < 0$$

$$n = n+1$$

$$x_1^{n+1} = x_1^n \ , \quad x_2^{n+1} = \frac{x_1^n + x_2^n}{2}$$

$$f(x_1^{n+1})f(x_2^{n+1}) < 0$$

yes

no

$$x_1^{n+1} = x_2^{n+1} \ , \quad x_2^{n+1} = x_2^n$$

bisect.m

```
% Root finding by bi-section
f=inline(' a*x -1','x','a');
a=2
figure(1); clf; hold on
x=[0 1.5]; eps=1e-3;
err=max(abs(x(1)-x(2)),abs(f(x(1),a)-f(x(2),a)));
while (err>eps & f(x(1),a)*f(x(2),a) <= 0)
    xo=x; x=[xo(1) 0.5*(xo(1)+xo(2))];
    if ( f(x(1),a)*f(x(2),a) > 0 )
        x=[0.5*(xo(1)+xo(2)) xo(2)]
    end
    x
    err=max(abs(x(1)-x(2)),abs(f(x(1),a)-f(x(2),a)));
    b=plot(x,f(x,a),'.b'); set(b,'MarkerSize',20);
    grid on;
end
```