Numerical Marine Hydrodynamics

- Interpolation
  - Lagrange interpolation
  - Triangular families
  - Newton's iteration method
  - Equidistant Interpolation
- Numerical Integration
  - Newton-Cotes Integration
    - Errors of numerical integration
  - Romberg Integration
    - Richardson Extrapolation
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  - Gaussian Quadrature
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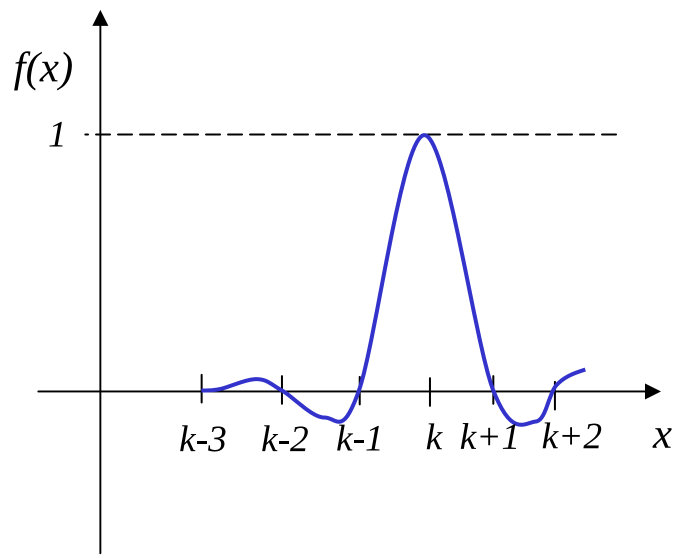
Numerical Interpolation
Lagrange Polynomials

$$p(x) = \sum_{k=0}^{n} L_k(x) f(x_k) = \sum_{k=0}^{n} L_k(x) f_k$$

$$L_k(x) = \sum_{i=0}^{n} \ell_{ik} x^i$$

$$L_k(x_i) = \delta_{ki} = \begin{cases} 0 & k \neq i \\ 1 & k = i \end{cases}$$

$$L_k(x) = \prod_{j=0, j\neq k}^{n} \frac{x - x_j}{x_k - x_j}$$



Difficult to program
Difficult to estimate errors
Divisions are expensive

Important for numerical integration

Numerical Integration

Lagrange Interpolation

$$I = \int_a^b f(x)dx$$

$$f(x) \simeq p(x) = \sum_{k=0}^{n} L_k(x) f(x_k)$$

$$L_k(x) = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

Equidistant Sampling

$$\begin{aligned} x_k &= x_0 + kh \\ x &= x_0 + sh \end{aligned}$$

$$L_k(x) = \frac{s(s-1)(s-2)\cdots(s-k+1)(s-k-1)\cdots(s-n)}{k(k-1)(k-2)\cdots(1)(-1)\cdots(k-n)}$$

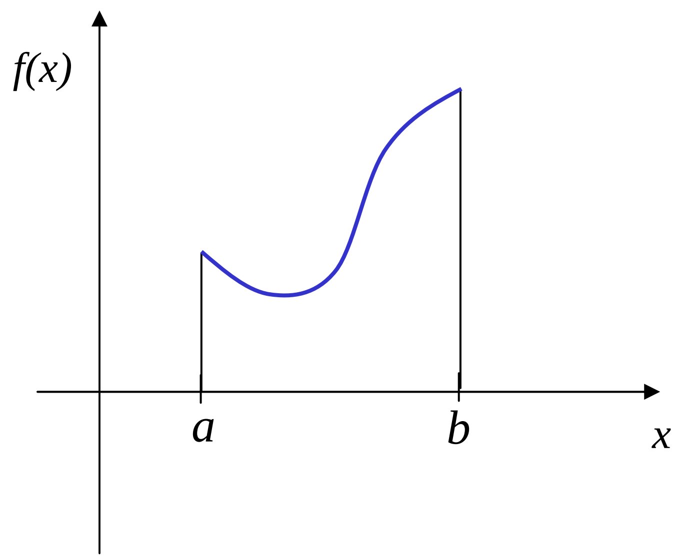
$$I = \int_a^b f(x)dx \simeq \int_{x_0}^{x_n} p(x)dx = h\sum_{k=0}^{n} f(x_k)\int_0^n L_k(s)ds = nh\sum_{k=0}^{n} f(x_k)C_k^n$$

Integration Weights (Cote's Numbers)

$$C_k^n = \frac{1}{n}\int_0^n L_k(s)ds$$

Properties

$$\begin{aligned} C_k^n &= C_{n-k}^n \\ \sum_{k=0}^{n} C_k^n &= 1 \end{aligned}$$



Numerical Integration

Trapezoidal Rule

$n = 1$

$$k=0: \quad C_0^1 = \int_0^1 \frac{s-1}{-1} ds = 1 - 1/2 = 0.5$$

$$k=1: \quad C_1^1 = \int_0^1 \frac{s}{1} ds = 1/2 = 0.5$$

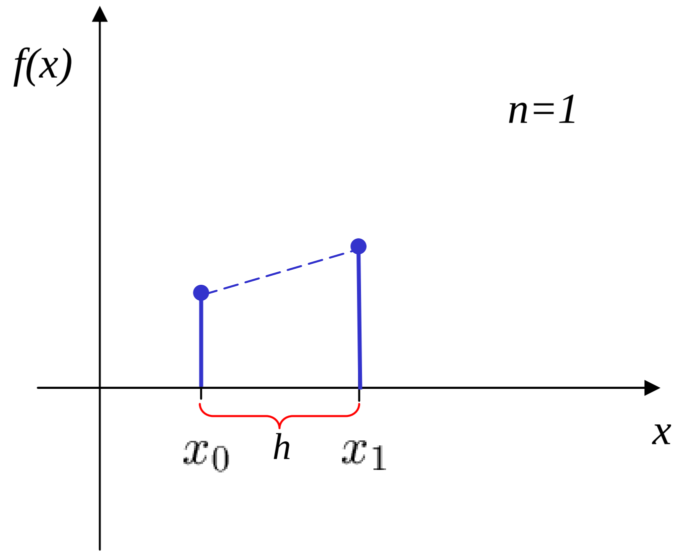
$$\int_{x_0}^{x_1} f(x)dx \simeq 1 \cdot (x_1 - x_0)\left(\frac{1}{2}f(x_0) + \frac{1}{2}f(x_1)\right) = \frac{1}{2}(x_1 - x_0)(f(x_0) + f(x_1))$$

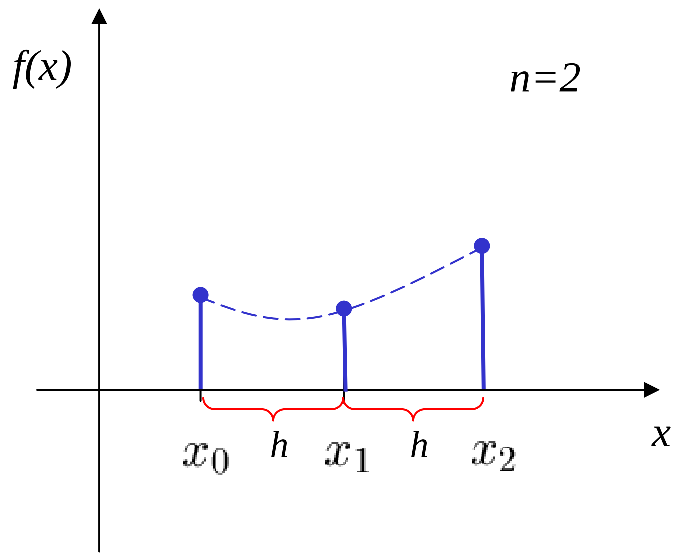
Simpson's Rule

$n = 2$

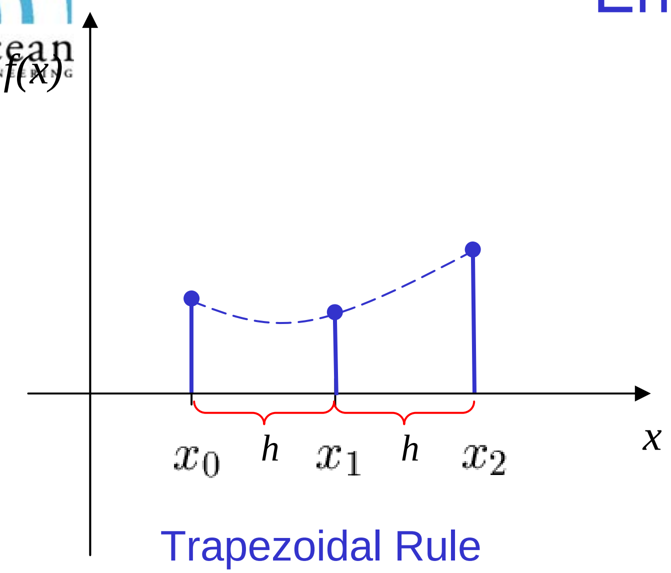
$$\begin{aligned}
k=0: \quad C_0^2 &= \tfrac{1}{2}\int_0^2 \tfrac{(s-1)(s-2)}{(-1)(-2)} ds \\
&= \tfrac{1}{4}\int_0^2 (s^2 - 3s + 2) ds \\
&= \tfrac{1}{4}\left[\tfrac{s^3}{3} - \tfrac{3s^2}{2} + 2s\right] \\
&= \tfrac{1}{4}\left[\tfrac{8}{3} - \tfrac{12}{2} + 4\right] = \tfrac{1}{4}\cdot\tfrac{4}{6} = \tfrac{1}{6} \\
k=1: \quad C_1^2 &= \tfrac{1}{2}\int_0^2 \tfrac{s(s-2)}{(1)(-1)} ds \\
&= \tfrac{1}{2}\int_0^2 (2s - s^2) ds \\
&= \tfrac{1}{2}\left[s^2 - \tfrac{s^3}{3}\right] \\
&= \tfrac{1}{2}\left[4 - \tfrac{8}{3}\right] = \tfrac{2}{3} \\
k=2: \quad C_2^2 &= C_0^2 = \tfrac{1}{6}
\end{aligned}$$

$$\int_{x_0}^{x_1} f(x)dx \simeq 2h\frac{1}{6}(f(x_0) + 4f(x_1) + f(x_2)) = \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2))$$





Numerical Integration
Error Analysis



Trapezoidal Rule

$$n = 1$$

$$e(x) = p(x) - f(x) = -\frac{f''(\xi)}{2}(x - x_0)(x - x_1)$$

Local Absolute Error

$$\begin{aligned} |\epsilon| &= \left| - \int_{x_0}^{x_1} \frac{f''(\xi)}{2}(x - x_0)(x - x_1)dx \right| \\ &\leq -\frac{\max|f''|}{2}(x - x_0)(x - x_1)dx \\ &= \frac{\max|f''|}{2} h^3 \int_0^1 s(s-1)ds = \frac{h^3}{12}\max|f''| \simeq O(h^3) \end{aligned}$$

N Intervals

$$E = \sum_{i=1}^{N} \epsilon_i \leq \frac{h^3}{12}\sum_{i=1}^{N}\max|f''| \leq \frac{Nh^3}{12}\max|f''| = \frac{(b-a)h^2}{12}\max|f''| \simeq O(h^2)$$

Simpson's Rule

$$n = 2$$

$$I = \int_{x_{m-1}}^{x_{m+1}} f(x)dx \simeq \frac{h^3}{3}[f_{m-1} + 4f_m + f_{m+1}]$$

Local Error

$$\epsilon_m = -\int_{x_{m-1}}^{x_{m+1}} \frac{f'''(\xi)}{6}(x - x_{m-1})(x - x_m)(x - x_{m+1})dx \simeq O(h^4)$$

Global Error

$$E = O(h^3)$$

$$x_m = 0,\ x_{m-1} = -h,\ x_{m+1} = h$$

$$f(x) = f_0 + xf_0' + \frac{x^2}{2}f_0'' + \frac{x^3}{3!}f_0''' + O(h^4)$$

$$\begin{aligned} I &= \int_{-h}^{h} f(x)dx \\ &= f_0 \int_{-h}^{h} xdx + \frac{f_0''}{2}\int_{-h}^{h} x^2dx + \frac{f_0'''}{6}\int_{-h}^{h} x^3dx + O(h^4) \\ &= 2hf_0 + 0 + \frac{h^3}{3}f_0'' + 0 + O(h^5) \end{aligned}$$

$$f_0'' = \frac{1}{h^2}(f_{-1} - 2f_0 + f_1) + O(h^2)$$

$$\begin{aligned} I &= 2hf_0 + \frac{h}{3}(f_{-1} - 2f_0 + f_1) + O(h^5) \\ &= \frac{h}{3}(f_{-1} + 4f_0 + f_1) + O(h^5) \end{aligned}$$

Local Error: $O(h^5)$

Global Error: $E = O(h^4)$

Romberg Integration
Richardson Extrapolation

Integration Error

$$I = I(h) + E(h)$$

$$I(h_1) + E(h_1) = I(h_2) + E(h_2)$$

Trapezoidal Rule

$$E(h) = -\frac{b-a}{12} h^2 \hat{f}''$$

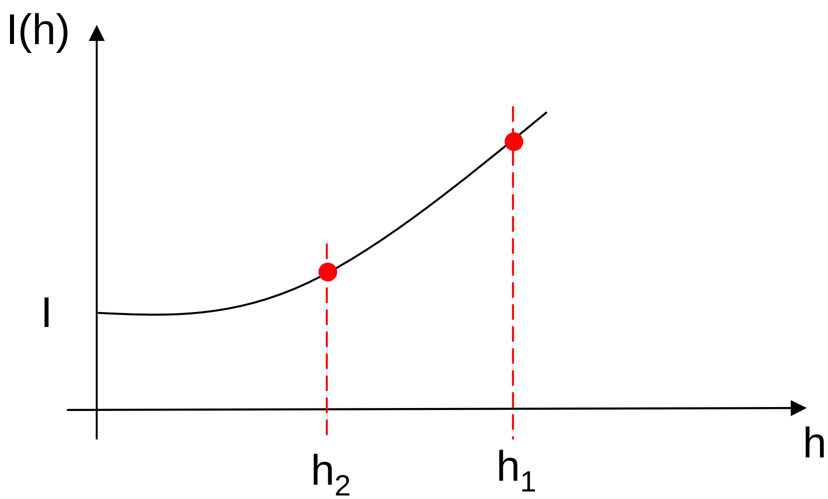
$$E(h_1) =\simeq E(h_2)\left(\frac{h_1}{h_2}\right)$$

$$I(h_1) + E(h_2)\left(\frac{h_1}{h_2}\right) \simeq I(h_2) + E(h_2)$$

$$E(h_2) \simeq \frac{I(h_1) - I(h_2)}{1 - (h_1/h_2)^2}$$

Richardson Extrapolation

$$I = I(h_2) + \frac{I(h_2) - I(h_1)}{(h_1/h_2) - 1} + O(h^4)$$

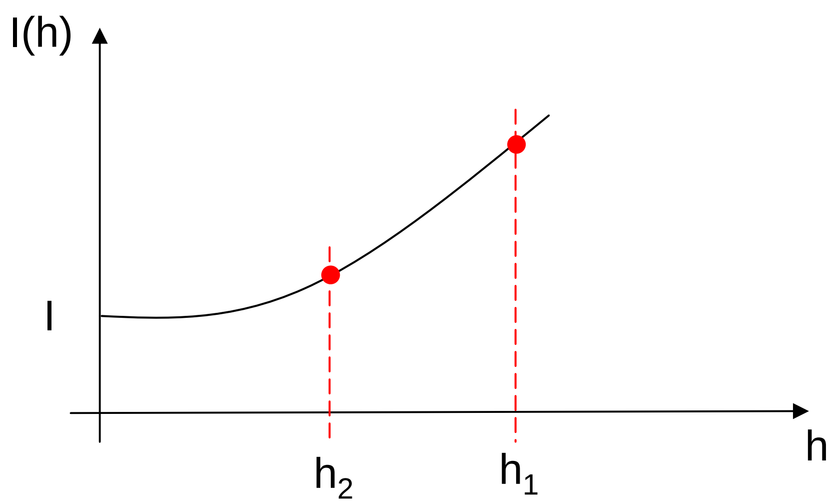


$$h_2 = h_1/2$$

$$I = I(h_2) + \frac{I(h_2) - I(h_1)}{(2^2 - 1} + O(h^4)$$

$$= \frac{4}{3} I(h_2) - \frac{1}{3} I(h_1) + O(h^4)$$

Romberg Integration



Romberg Integration Algorithm

$$I_{j,k} \simeq \frac{4^{k-1} I_{j+1,k-1} - I_{j,k-1}}{4^{k-1} - 1}$$

Order 2

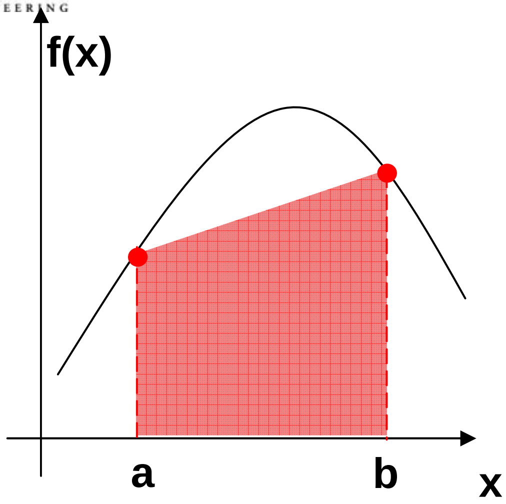
$$k = 2, j = 1$$

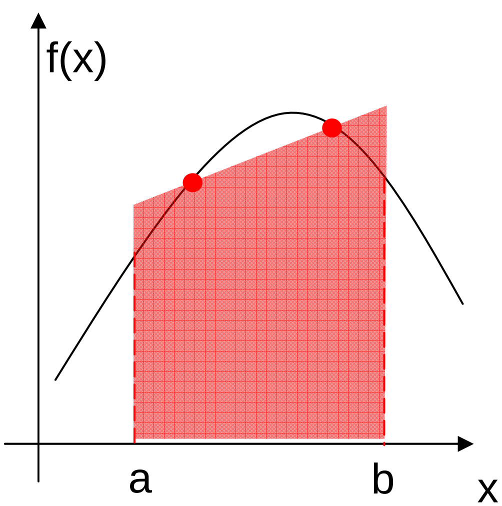
$$I_{1,2} \simeq \frac{4I_{2,1} - I_{1,1}}{3}$$

k →

	1: $O(h^2)$	2: $O(h^4)$	3: $O(h^6)$	4: $O(h^8)$
a.	0.172800	1.367467		
	1.068800			
b,.	0.172800	1.367467	1.640533	
	1.068800	1.623467		
	1.484800			
c.	0.172800	1.367467	1.640533	1.640533
j ↓	1.068800	1.623467	1.640533	
	1.484800	1.639467		
	1.600800			

Gaussian Quadrature





Trapezoidal Rule

$$I \simeq (b-a)\frac{f(a)+f(b)}{2}$$

$$I \simeq c_0 f(a) + c_1 f(b)$$

Exact Integration of const and linear f

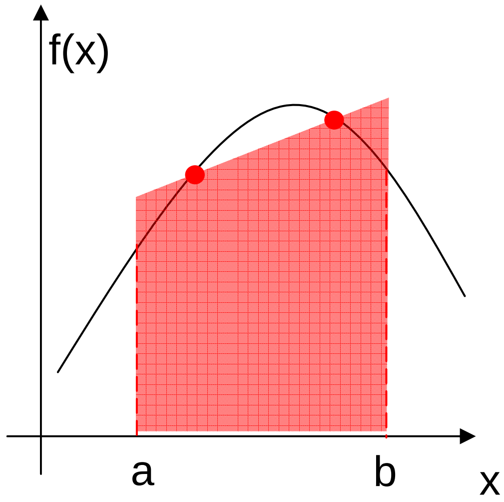
$$c_0 + c_1 = \int_{-(b-a)/2}^{-(b-a)/2} 1dx$$

$$-c_0\frac{b-a}{2} + c_1\frac{b-a}{2} = \int_{-(b-a)/2}^{-(b-a)/2} xdx$$

$$\begin{aligned} c_0 + c_1 &= b-a \\ -c_0\frac{a-b}{2} + c_1\frac{a-b}{2} &= 0 \\ &\Rightarrow \\ c_0 = c_1 &= \frac{b-a}{2} \end{aligned}$$

$$I \simeq \frac{b-a}{2}f(a) + \frac{b-a}{2}f(b)$$

Gaussian Quadrature



$$a = -1\ ,\ \ b = 1$$

$$I = c_0 f(x_0) + c_1 f(x_1)$$

$$\begin{aligned} c_0 + c_1 &= \int_{-1}^{1} 1 dx = 2 \\ c_0 x_0 + c_1 f x_1 &= \int_{-1}^{1} x dx = 0 \\ c_0 x_0^2 + c_1 x_1^2 &= \int_{-1}^{1} x^2 dx = 2/3 \\ c_0 x_0^3 + c_1 x_1^3 &= \int_{-1}^{1} x_3 dx = 0 \end{aligned}$$

$$\begin{aligned} c_0 &= 1 \\ c_1 &= 1 \\ x_0 &= -\frac{1}{\sqrt{3}} \\ x_1 &= \frac{1}{\sqrt{3}} \end{aligned}$$

2-point Gaussian Quadrature

$$I \simeq f(\frac{-1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}})$$