9 Fluid dynamics and Rayleigh-Bénard convection

In these lectures we derive (mostly) the equations of viscous fluid dynamics. We then show how they may be generalized to the problem of Rayleigh-Bénard convection—the problem of a fluid heated from below. Later we show how the RB problem itself may be reduced to the famous Lorenz equations.

The highlights of these lectures are as follows:

- Navier-Stokes equations of fluid dynamics (mass and momentum conservation).
- Reynolds number
- Phenomenology of RB convection
- Rayleigh number
- Equations of RB convection

Thus far we have dealt almost exclusively with the temporal behavior of a few variables.

In these lectures we digress, and discuss the evolution of a *continuum*.

9.1 The concept of a continuum

Real fluids are made of atoms or molecules.

The *mean free path* ℓ_{mfp} is the characteristic length scale between molecular collisions.

Let L_{hydro} be the characteristic length scale of macroscopic motions.

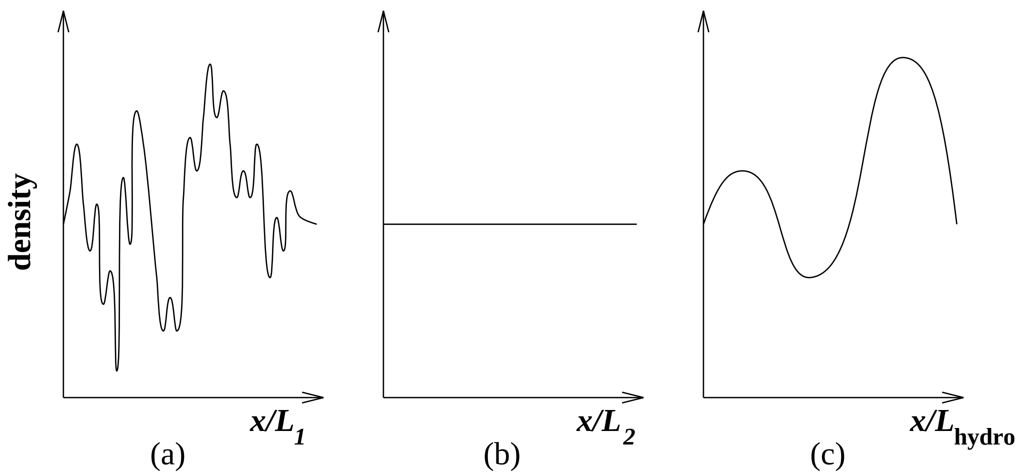
Fluids may be regarded as *continuous fields* if

$$L_{\text{hydro}} \gg \ell_{\text{mpf}}.$$

When this condition holds, the evolution of the macroscopic field may be described by *continuum mechanics*, i.e., partial differential equations.

To make this idea clearer, consider a thought experiment in which we measure the density of a fluid over a length scale ℓ using some particularly sensitive device. We then move the device in the x-direction over a distance of roughly 10ℓ.

Suppose $\ell \sim L_1 \sim \ell_{mpf}$. Then we expect the density to vary greatly in space as in Figure (a) below:



We expect that the fluctuations in (a) should decrease as ℓ increases. (Statistics tells us that these fluctuations should decrease like $1/N^{1/2}$, where $N \propto \ell^3$ is the average number of molecules in a box of size ℓ.)

On the other hand, if $\ell \sim L_{\rm hydro}$ (see (c)), variations in density should reflect density changes due to macroscopic motions (e.g., a rising hot plume), not merely statistical fluctuations.

Our assumption of a continuum implies that there is an intermediate scale, $\ell \sim L_2$, over which fluctuations are small. Thus the continuum hypothesis implies a *separation of scales* between the molecular scale, $L_1 \sim \ell_{\rm mfp}$, and the hydrodynamic scale, $L_{\rm hydro}$.

Thus, rather than dealing with the motion $\sim 10^{23}$ molecules and therefore $\sim 6 \times 10^{23}$ ordinary differential equations of motion (3 equations each for position and momentum), we model the fluid as a continuum.

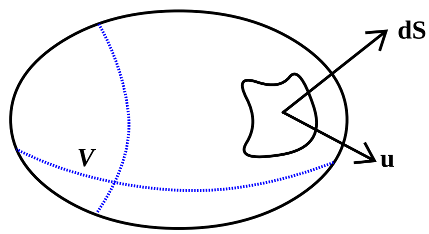
The motion of the continuum is expressed by *partial differential equations* for evolution of conserved quantities. We begin with the conservation of mass.

9.2 Mass conservation

Let

$$\left.\begin{array}{l}\rho = \text{density}\\ \vec{u} = \text{velocity}\end{array}\right\} \text{ of a macroscopic fluid particle}$$

Consider a volume V of fluid, fixed in space:



$\mathrm{d}\vec{s}$ is an element of the surface, $|\mathrm{d}\vec{s}|$ is its area, and it points in the outward normal direction.

$\vec{u}$ is the velocity.

The outward mass flux through the element $\mathrm{d}\vec{s}$ is

$$\rho\vec{u}\cdot\mathrm{d}\vec{s}.$$

Therefore,

$$\text{rate of mass loss from } V = \int_s \rho\vec{u}\cdot\mathrm{d}\vec{s}.$$

The total mass in V is

$$\int_V \rho\mathrm{d}v$$

Thus the rate of mass loss may be rewritten as

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\mathrm{d}v = -\int_V \frac{\partial\rho}{\partial t}\mathrm{d}v = +\int_s \rho\vec{u}\cdot\mathrm{d}\vec{s}$$

Shrinking the volume, we eliminate the integrals and obtain

$$\frac{\partial\rho}{\partial t} = -\lim_{V\to 0}\left[\int \rho\vec{u}\cdot\mathrm{d}\vec{s}/V\right].$$

Recall that the RHS above is the definition of the divergence operator. We thus obtain

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot (\rho \vec{u})$$

We see that to conserve mass, a net divergence creates a corresponding change in density.

For *incompressible* fluids,

$$\rho \sim \text{constant.}$$

(This result is not an assumption, but instead derives from the assumption that the Mach number, the square of the ratio of the fluid velocity to the speed of sound, is much less than unity.)

Then

$$\vec{\nabla} \cdot \vec{u} = 0.$$

which is the *equation of continuity* for incompressible fluids.

9.3 Momentum conservation

We seek an expression of Newton's second law:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{momentum of fluid particle}) = \text{force acting on fluid particle} \tag{22}$$

9.3.1 Substantial derivative

We first focus on the LHS of (22).

There is a conceptual problem: $\frac{\mathrm{d}}{\mathrm{d}t}$(particle momentum) cannot be given at a fixed location, because

- the momentum field itself changes with respect to time; and
- fluid particle can change its momentum by *flowing* to a place where the velocity is different.

To better understand this problem physically, consider how a scalar property—the temperature T—of a fluid particle changes in time.

A small change δT is produced by a small changes δt in time and δx, δy, δz in the position of the fluid particle:

$$\delta T = \frac{\partial T}{\partial t}\delta t + \frac{\partial T}{\partial x}\delta x + \frac{\partial T}{\partial y}\delta y + \frac{\partial T}{\partial z}\delta z$$

Divide by δt to obtain the rate of change:

$$\frac{\delta T}{\delta t} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial T}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial T}{\partial z}\frac{\delta z}{\delta t}$$

In the limit $\delta t \to 0$,

$$\frac{\delta x}{\delta t} \to u_x, \qquad \frac{\delta y}{\delta t} \to u_y, \qquad \frac{\delta z}{\delta t} \to u_z$$

The rate of change of T of a fluid particle is then

$$\begin{aligned} \frac{\mathrm{D}T}{\mathrm{D}t} &= \frac{\partial T}{\partial t} + u_x\frac{\partial T}{\partial x} + u_y\frac{\partial T}{\partial y} + u_z\frac{\partial T}{\partial z} \\ &= \frac{\partial T}{\partial t} + \vec{u}\cdot\vec{\nabla}T \end{aligned}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}$$

is the *substantial derivative* or *convective derivative* operator.

Thus we see that the temperature of a fluid particle can change because

- the temperature field changes "in place" (via $\partial/\partial t$); and
- the particle can flow to a position where the temperature is different (via $\vec{u}\cdot\vec{\nabla}$).

Note that the same analysis applies to *vector* fields such as the velocity $\vec{u}$:

$$\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = \frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}$$

Therefore the velocity $\vec{u}$ enters $\mathrm{D}\vec{u}/\mathrm{D}t$ in 2 ways:

- $\vec{u}$ changes (in place) as the fluid moves $(\partial/\partial t)$
- $\vec{u}$ governs *how fast* that change occurs $(\vec{u} \cdot \vec{\nabla})$.

This dual role of velocity is the essential nonlinearity of fluid dynamics and thus the cause of turbulent instabilities.

We can now express the rate-of-change of momentum per unit volume (i.e., LHS of (22)):

$$\rho \frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = \rho \frac{\partial \vec{u}}{\partial t} + \rho(\vec{u} \cdot \vec{\nabla})\vec{u}$$

ρ is outside the differential because a fluid particle does not lose mass. Density changes thus mean volume changes, which are irrelevant to the momentum change of that particle. Above we have written the (rate of change of momentum) per unit volume, which need not be equal to the rate of change of (momentum per unit volume).

9.3.2 Forces on fluid particle

To obtain the full dynamical equation, we need the RHS of

$$\rho \frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = \text{ Force acting on fluid particle / unit volume.}$$

These forces are

- body force (i.e., gravity)
- pressure
- viscous friction (internal stresses)

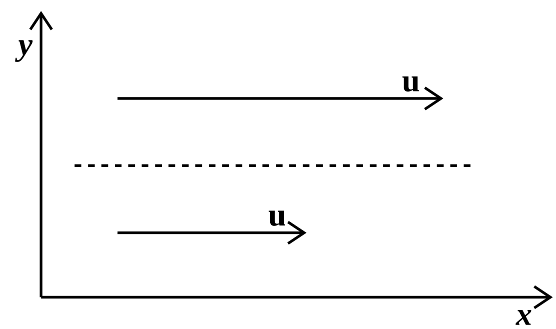
Body force. We represent the externally imposed body force by $\vec{F}$.

Pressure. Fluid flows from high to low pressure. Thus

$$\begin{aligned}\frac{\text{pressure force}}{\text{unit volume}} &= -\frac{\partial p}{\partial x} \qquad \text{in 1-D} \\ &= -\vec{\nabla} p \qquad \text{in 3-D}\end{aligned}$$

Viscous friction. Viscous stresses are the source of dissipation in fluids. They resist relative movements between fluid particles.

For example, the shear flow



is resisted more by high viscosity fluids than low viscosity fluids.

This resistance derives from molecular motions. (A nice analog is Reif's picture of two mail trains, one initially fast and the other initially slow, that trade mailbags.)

In the simple shear flow above, there is a flux of x-momentum in the y-direction.

In *Newtonian fluids*, this flux, which we call P_{xy}, is proportional to the gradient:

$$P_{xy} = -\eta \frac{\partial u_x}{\partial y}$$

where η is called the dynamic viscosity. η has units of mass/(length $\times$ time).

The shear stress can occur at any orientation. Analogous to the 1-D Newtonian condition above, we define the *viscous momentum flux*

$$P_{ij} = -\eta \frac{\partial u_i}{\partial x_j}.$$

The conservation of momentum requires that the divergence of the momentum flux P_{ij} be balanced by a change in the momentum of a fluid particle. Loosely stated,

$$\left.\frac{\partial(\rho u_i)}{\partial t}\right|_{\text{viscous}} = -\vec{\nabla}\cdot P_{ij} = -\sum_j \frac{\partial}{\partial x_j}P_{ij} = \eta\sum_j \frac{\partial^2}{\partial x_j^2}u_i$$

We thus find that

$$\frac{\text{viscous force}}{\text{unit volume}} = \eta\nabla^2\vec{u}.$$

(A careful derivation requires consideration of the tensorial relationship between viscous stress and the rate of deformation.)

Newton's second law then gives the *Navier-Stokes equation* for incompressible fluids:

$$\underbrace{\rho\frac{\partial\vec{u}}{\partial t} + \rho(\vec{u}\cdot\vec{\nabla})\vec{u}}_{\text{(mass per unit vol)}\times\text{acceleration}} \quad = \quad \underbrace{-\vec{\nabla}p + \eta\nabla^2\vec{u}}_{\text{stresses on fluid element per unit vol}} \quad + \quad \underbrace{\vec{F}}_{\text{body force per unit vol}}$$

Incompressibility arose from our negelect of compressive forces on fluid elements.

9.4 Nondimensionalization of Navier-Stokes equations

Define the characteristic length scale L and velocity scale U. We obtain the non-dimensional quantities

$$x' = \frac{x}{L}, \quad y' = \frac{y}{L}, \quad z' = \frac{z}{L}$$

$$\vec{u}' = \frac{\vec{u}}{U}, \quad t' = t\frac{U}{L}, \quad p' = \frac{p}{\rho U^2}$$

The dynamical equations (without body force) become

$$\vec{\nabla}'\cdot\vec{u}' = 0$$

$$\frac{\partial\vec{u}'}{\partial t'} + (\vec{u}'\cdot\vec{\nabla}')\vec{u}' = -\vec{\nabla}'p' + \frac{1}{\text{Re}}\nabla'^2\vec{u}'$$

where

$$\mathrm{Re} = \text{ Reynolds number } = \frac{\rho U L}{\eta}$$

is the *dimensionless* control parameter.

The Reynolds number quantifies the relative importance of the nonlinear term to the viscous term. To see why, note the following dimensional quantities:

$$|\rho\vec{u}\cdot\vec{\nabla}\vec{u}| \sim \frac{\rho U^2}{L} \qquad \text{nonlinearity}$$

$$|\eta\nabla^2\vec{u}| \sim \frac{\eta U}{L^2} \qquad \text{dissipation}$$

Their ratio is

$$\frac{|\rho\vec{u}\cdot\vec{\nabla}\vec{u}|}{|\eta\nabla^2\vec{u}|} \sim \frac{\rho U L}{\eta} = \text{Reynolds number}$$

High Re is associated with turbulence (i.e., nonlinearities). Low Re is associated with *laminar* or *creeping* flows dominated by viscous friction.

Note that as long as Re remains the same, the dimensional parameters like U and L can change but the the flow (i.e., the equation it solves) does not. This is *dynamical similarity.*

An example is running vs. swimming:

$$\left.\left(\frac{\eta}{\rho}\right)\right|_{\text{air}} = 0.15\ \text{cm}^2/\text{sec} \qquad \text{and} \qquad \left.\left(\frac{\eta}{\rho}\right)\right|_{\text{water}} = 0.01\ \text{cm}^2/\text{sec}$$

On the other hand, comparing 100 meter world records,

$$U_{\text{run}} \sim \frac{10^4\ \text{cm}}{10\ \text{sec}} = 10^3\ \text{cm/sec}$$

$$U_{\text{swim}} \sim \frac{10^4\ \text{cm}}{55\ \text{sec}} \sim 2\times 10^2\ \text{cm/sec}$$

Taking $L \sim 100$ cm,

$$\mathrm{Re(swim)} \sim 2\times 10^4 \qquad \text{and} \qquad \mathrm{Re(run)} \sim 6\times 10^3$$

Thus for both swimming and running, Re $\sim 10^4$, well into the turbulent regime. Surprisingly, despite the slower speed of swimming, Re(swim) is somewhat greater.

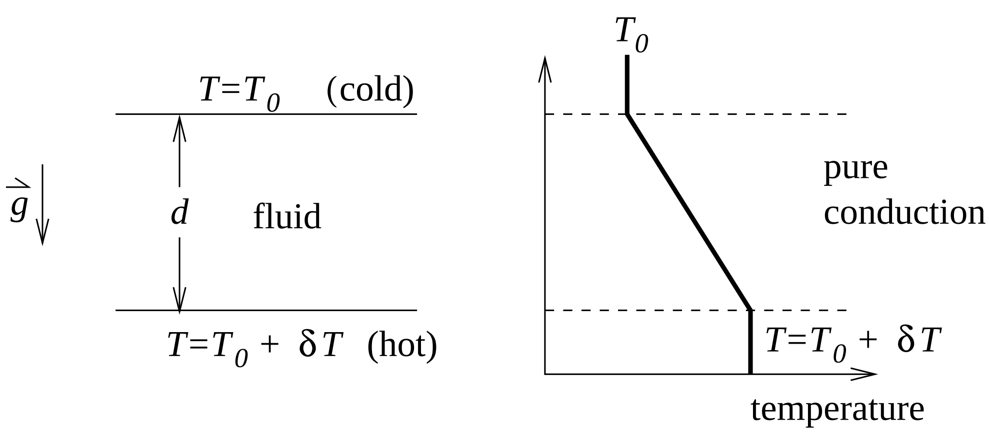
Another example: bacteria swimming in water is roughly like us swimming in molasses, since the the small size and slow speed of bacteria would correspond to a larger and faster body in a more viscous fluid.

9.5 Rayleigh-Bénard convection

In a thermally expansive fluid, hot fluid rises.

R-B convection concerns the study of the instabilities caused by rising hot fluid and falling cold fluid.

Typically,, fluid is confined between two horizontal, heat-conducting plates:



In the absence of convection—the transport of hot fluid *up* and cold fluid *down*—the temperature gradient is constant.

Two cases of interest:

- δT small: no convective motion, due to stabilizing effects of viscous friction.
- δT large: convective motion occurs.

How large is a "large δT" ? We seek a non-dimensional formulation.

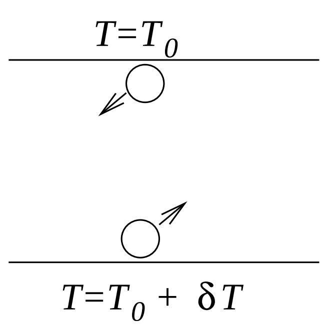
The following fluid properties are important:

- viscosity
- density
- thermal expansivity
- thermal diffusivity (heat conductivity)

Convection is also determined by

- d, the box size
- δT (of course)

Consider a small displacement of a cold blob downwards and a hot blob upwards:



Left undisturbed, buoyancy forces would allow the hot blob to continue rising and cold blob to continue falling.

There are however damping (dissipation) mechanisms:

- diffusion of heat
- viscous friction

Let D_T = thermal diffusivity, which has units

$$[D_T] = \frac{\text{length}^2}{\text{time}}$$

The temperature difference between the two blobs can therefore be maintained at a characteristic time scale

$$\tau_{\text{th}} \sim \frac{d^2}{D_T}$$

We also seek a characteristic time scale for buoyant displacement over the length scale d.

Let

$$\begin{aligned} \rho_0 &= \text{mean density} \\ \Delta\rho &= -\alpha\rho_0\Delta T, \qquad \alpha = \text{expansion coefficient} \end{aligned}$$

Setting $\Delta T = \delta T$,

$$\begin{aligned} \text{buoyancy force density} &= |\vec{g}\Delta\rho| \\ &= g\alpha\rho_0\,\delta T. \end{aligned}$$

Note units:

$$[g\alpha\rho_0\delta T] = \frac{\text{mass}}{(\text{length})^2(\text{time})^2}$$

The buoyancy force is resisted by viscous friction between the two blobs separated by $\sim d$.

The viscous friction between the two blobs diminishes like $1/d$ (since viscous stresses $\propto$ velocity gradients). The rescaled viscosity has units

$$\left[\frac{\eta}{d}\right] = \frac{\text{mass}}{(\text{length})^2(\text{time})}$$

Dividing the rescaled viscosity by the buoyancy force, we obtain the characteristic time τ_m for convective motion:

$$\tau_m \sim \frac{\eta/d}{\text{buoyancy force}} = \frac{\eta}{g\alpha\rho_0\,d\,\delta T}.$$

Convection (sustained motion) occurs if

$$\begin{aligned} \text{time for motion} &< \text{diffusion time for temperature difference} \\ \tau_{\mathrm{m}} &< \tau_{\mathrm{th}} \end{aligned}$$

Thus convection requires

$$\frac{\tau_{\mathrm{th}}}{\tau_{\mathrm{m}}} > \text{constant}$$

or

$$\frac{\rho_0 g \alpha d^3}{\eta D_T} \delta T \equiv \mathrm{Ra} > \text{constant}$$

Ra is the *Rayleigh number*. A detailed stability calculation reveals that the critical constant is 1708.

Our derivation of the Rayleigh number shows that the convective instability is favored by

- large δT, α, d, ρ_0.
- small η, D_T.

In other words, convection occurs when the buoyancy force $\rho_0 g \alpha d^3 \, \delta T$ exceeds the dissipative effects of viscous drag and heat diffusion.

Note that box height enters Ra as d^3. This means that small increases in box size can have a dramatic effect on Ra.

9.6 Rayleigh-Bénard equations

9.6.1 Dimensional form

We employ the *Boussinesq approximation*: density perturbations affect only the gravitational force.

The momentum equation is therefore the Navier-Stokes equation augmented by the buoyancy force:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} = -\frac{1}{\rho_0} \vec{\nabla} p + \nu \nabla^2 \vec{u} - \vec{g} \alpha (T - T_0)$$

Here we have written the *kinematic viscosity*

$$\nu = \eta/\rho_0$$

The mass conservation equation is again

$$\vec{\nabla} \cdot \vec{u} = 0.$$

We now additionally require an equation for the convection and diffusion of heat:

$$\frac{\partial T}{\partial t} + (\vec{u} \cdot \nabla)T = D_T \nabla^2 T.$$

9.6.2 Dimensionless equations

The equations are nondimensionalized using

$$\begin{aligned} \text{length scale} &= d \\ \text{time scale} &= d^2/D_T \\ \text{temperature scale} &= \delta T/\mathrm{Ra}. \end{aligned}$$

An additional dimensionless parameter arises:

$$\mathrm{Pr} = \text{Prandtl number} = \nu/D_T,$$

which is like the ratio of momentum diffusion to thermal diffusion.

We shall employ the dimensionless temperature fluctuation

$$\theta = \text{deviation of dimensionless } T \text{ from the simple conductive gradient}$$

The mass conservation equation is

$$\vec{\nabla} \cdot \vec{u} = 0$$

Momentum conservation yields ($\hat{z}$ is a unit upward normal)

$$\frac{1}{\mathrm{Pr}} \left[\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} \right] = -\vec{\nabla} p + \theta \hat{z} + \nabla^2 \vec{u}$$

The heat equation becomes

$$\frac{\partial \theta}{\partial t} + \vec{u} \cdot \vec{\nabla}\theta = \mathrm{Ra}(\vec{u} \cdot \hat{z}) + \nabla^2\theta$$

Note that there are two nonlinear terms:

- $\vec{u} \cdot \vec{\nabla}\vec{u}$
- $\vec{u} \cdot \vec{\nabla}\theta$

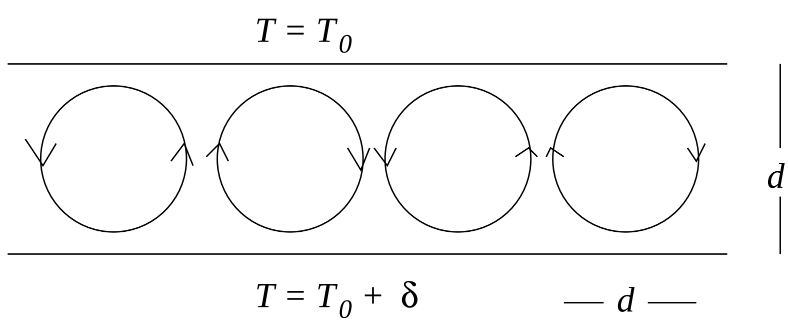
Their relative importance depends on Pr:

- small Pr $\Rightarrow \vec{u} \cdot \vec{\nabla}\vec{u}$ dominates. Instabilities are "hydrodynamic."
- large Pr $\Rightarrow \vec{u} \cdot \vec{\nabla}\theta$ dominates. Instabilities are thermally induced.

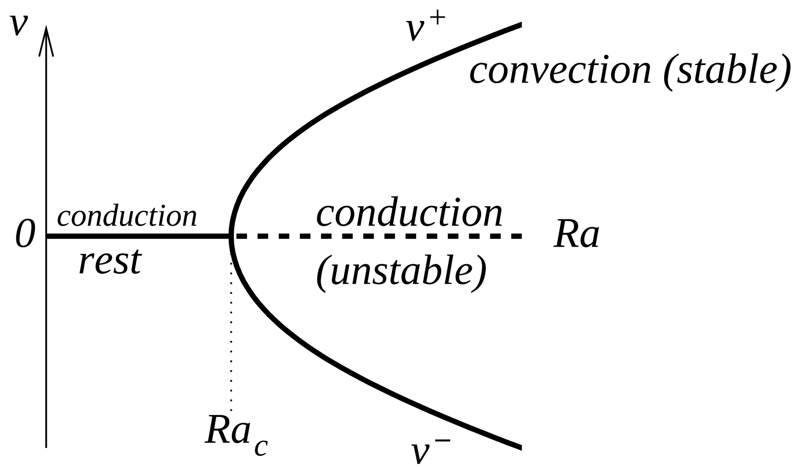
9.6.3 Bifurcation diagram

For Ra $<$ Ra$_c$, there is no convection.

For Ra $>$ Ra$_c$, but not too large, a regular structure of convection "rolls" forms, with hot fluid rising and cold fluid falling:



Now imagine placing a probe that measures the vertical component v of velocity, somewhere in the box midway between the top and bottom. A plot of $v(\mathrm{Ra})$ looks like



Such a plot is called a *bifurcation* diagram. Here the stable states are bold and the unstable states are dashed.

Note that we cannot know in advance whether the velocity will be up or down. This is called *symmetry breaking*.

9.6.4 Pattern formation

Rayleigh-Bénard convection makes fascinating patterns. Some examples:

- Figures 22.3–8, Tritton.
- Plate 1, Schuster, showing quasiperiodic regime. (The 40 sec period is not precise: note details in upper right are not quite periodic.)
- Plumes: Figure 22.12, Tritton
- Plumes in the wind: Zocchi, Moses, and Libchaber (1990)
- Collective plumes: Zhang et al (1997).

9.6.5 Convection in the Earth

The Earth's radius is about 6378 km. It is layered, with the main divisions being the inner core, outer core, mantle, and crust.

The Earth's crust—the outermost layer—is about 30 km thick.

The mantle ranges from about 30–2900 km.

The mantle is widely thought to be in a state of thermal convection. The source of heat is thought to be the radioactive decay of isotopes of uranium, thorium, and potassium. Another heat source is related to the heat deriving from the gravitational energy dissipated by the formation of the Earth roughly 4.5 Ga.

At long time scales mantle rock is thought to flow like a fluid. However its effective viscosity is the subject of much debate.

One might naively think that the huge viscosity would make the Rayleigh number quite small. Recall, however, that Ra scales like d^3, where d is the "box size". For the mantle, d is nearly 3000 km!!!

Consequently Ra is probably quite high. Current estimates suggest that

$$3 \times 10^6 \lesssim \mathrm{Ra}_{\mathrm{mantle}} \lesssim 10^9$$

which corresponds to roughly

$$10^3 \times \mathrm{Ra}_c \lesssim \mathrm{Ra}_{\mathrm{mantle}} \lesssim 10^6 \mathrm{Ra}_c$$

The uncertainty derives principally from the viscosity, and its presumed variation by a factor of about 300 with depth.

Some pictures illustrate these ideas:

- Science cover, 26 May 1989
- van der Hilst seismic tomography, showing cold slab descending toward the core-mantle boundary.
- Gurnis (1988) simulation/cartoon showing breakup of continents.
- Zhang and Libchaber (2000) showing floater-plates.

Thermal convection is the "engine" that drives plate tectonics and volcanism.

It turns out that volcanism is, over the long-term, responsible for the CO_2 in the atmosphere, and thus the source of carbon that is fixed by plants. (Weathering reactions remove C from the atmosphere.)

Thus in some sense thermal convection may be said to also sustain life.

That is, without convection, there probably would be no CO_2 in the atmosphere, and therefore we wouldn't be around to discuss it...