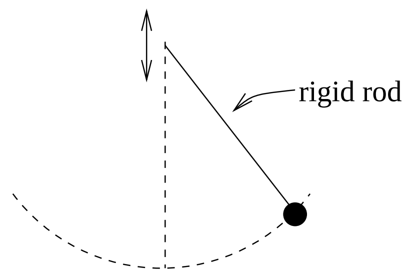
6 Parametric oscillator

6.1 Mathieu equation

We now study a different kind of forced pendulum.

Specifically, imagine subjecting the pivot of a simple frictionless pendulum to an alternating vertical motion:



This is called a "parametric pendulum," because the motion depends on a time-dependent parameter.

Consider the parametric forcing to be a time-dependent gravitational field:

$$g(t) = g_0 + \beta(t)$$

The linearized equation of motion is then (in the undamped case)

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g(t)}{l}\theta = 0.$$

The time-dependence of $g(t)$ makes the equation hard to solve. We know, however, that the rest state

$$\theta = \dot{\theta} = 0$$

is a solution. But is the rest state stable?

We investigate the stability of the rest state for a special case: $g(t)$ periodic and sinusoidal:

$$g(t) = g_0 + g_1 \cos(2\omega t)$$

Substituting into the equation of motion then gives

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega_0^2 \left[1 + h\cos(2\omega t)\right]\theta = 0 \tag{14}$$

where

$$\omega_0^2 = g_0/l \qquad \text{and} \qquad h = g_1/g_0 \gg 0.$$

Equation (14) is called the *Mathieu equation.*

The excitation (forcing) term has amplitude h and period

$$T_{\text{exc}} = \frac{2\pi}{2\omega} = \frac{\pi}{\omega}$$

On the other hand, the natural, unexcited period of the pendulum is

$$T_{\text{nat}} = \frac{2\pi}{\omega_0}$$

We wish to characterize the stability of the rest state. Our previous methods are unapplicable, however, because of the time-dependent parametric forcing.

We pause, therefore, to consider the theory of linear ODE's with periodic coefficients, known as *Floquet theory.*

6.2 Elements of Floquet Theory

Reference: Bender and Orszag, p. 560.

We consider the general case of a second-order linear ODE with periodic coefficients. We seek to determine the conditions for stability.

We begin with two observations:

1. If the coefficients are periodic with period T, then if $\theta(t)$ is a solution, so is $\theta(t+T)$.
2. Any solution $\theta(t)$ is a linear combination of two linearly independent solutions $\theta_1(t)$ and $\theta_2(t)$:
$$\theta(t) = A\theta_1(t) + B\theta_2(t) \tag{15}$$
where A and B come from initial conditions. (Reason: the system is linear and second-order.)

Since $\theta_1(t)$ and $\theta_2(t)$ are solutions, periodicity $\Rightarrow \theta_1(t+T)$ and $\theta_2(t+T)$ are also solutions.

Then $\theta_1(t+T)$ and $\theta_2(t+T)$ may themselves be represented as linear combinations of $\theta_1(t)$ and $\theta_2(t)$:

$$\begin{aligned} \theta_1(t+T) &= \alpha\theta_1(t) + \beta\theta_2(t) \\ \theta_2(t+T) &= \gamma\theta_1(t) + \delta\theta_2(t) \end{aligned}$$

Thus

$$\begin{aligned} \theta(t+T) &= A\left[\alpha\theta_1(t) + \beta\theta_2(t)\right] + B\left[\gamma\theta_1(t) + \delta\theta_2(t)\right] \\ &= (A\alpha + B\gamma)\theta_1(t) + (A\beta + B\delta)\theta_2(t) \end{aligned}$$

We rewrite the latter expression as

$$\theta(t+T) = A'\theta_1(t) + B'\theta_2(t) \tag{16}$$

where

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

or

$$\vec{a}' = M\vec{a}, \qquad \vec{a} = \begin{pmatrix} A \\ B \end{pmatrix}.$$

Now choose $\begin{pmatrix} A \\ B \end{pmatrix}$ to be an eigenvector of M, with λ the associated eigenvalue. Any other $\vec{a}$ would have a projection onto one of the eigenvectors.

Then

$$A' = \lambda A \qquad \text{and} \qquad B' = \lambda B.$$

Using (16) and (15), we find that

$$\theta(t+T) = \lambda\theta(t).$$

Thus $\theta(t)$ is periodic within a scale factor λ. The question of stability then hinges on the magnitude of λ.

Define

$$\mu = \frac{\ln|\lambda|}{T} \qquad \Rightarrow \qquad \lambda = e^{\mu T}.$$

(We are interested only in growth or decay, not oscillations due to the exponential multiplier.)

Then

$$\theta(t+T) = e^{\mu T}\theta(t).$$

Here θ is rescaled by $e^{\mu T}$ each period.

Finally, define $P(t)$ to be periodic such that $P(t+T) = P(t)$. The rescaling may now be expressed continuously as

$$\theta(t) = e^{\mu t}P(t).$$

[The above $\Rightarrow \theta(t+T) = e^{\mu(t+T)}P(t+T) = e^{\mu T}e^{\mu t}P(t) = e^{\mu T}\theta(t)$.]

Stability thus rests on the sign of μ.

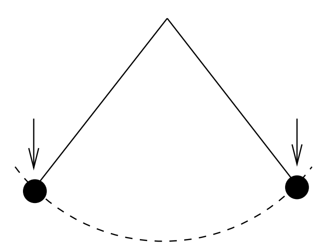
Thus we find that the solution to a linear second-order ODE with periodic coefficient—e.g., the Mathieu equation—is of the form

$$\text{(exponential growth or decay)} * \text{(periodic function of time)}$$

6.3 Stability of the parametric pendulum

We proceed to determine under what conditions the rest state of the Mathieu equation is unstable, leading to exponentially growing oscillations.

We expect the tendency toward instability to be strongest when the excitation frequency is twice the natural frequency, ω_0, of the pendulum.



This is called *parametric resonance*.

The Mathieu equation (for $\theta \ll 1$) is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega_0^2 \left[1 + h\cos(2\omega t)\right]\theta = 0.$$

We take the forcing frequency to be $2\omega_0 + \varepsilon$, $\varepsilon \ll 1$:

$$\ddot{\theta} + \omega_0^2 \left[1 + h\cos(2\omega_0 + \varepsilon)t\right]\theta = 0. \tag{17}$$

We assume that the excitation amplitude $h \ll 1$, and seek solutions of the general form

$$\theta(t) = a(t)\cos\left(\omega_0 + \frac{1}{2}\varepsilon\right)t + b(t)\sin\left(\omega_0 + \frac{1}{2}\varepsilon\right)t.$$

The frequency $\omega_0 + \frac{1}{2}\varepsilon$ is motivated by our resonance argument. Stability will depend on whether $a(t)$ and $b(t)$ exponentially grow or decay.

We begin by substituting $\theta(t)$ into the equation of motion. We will then retain terms that are linear in ε and first-order in h.

The calculation is messy but is aided by recalling trig identities like*

$$\cos\left[\left(\omega_0 + \frac{1}{2}\varepsilon\right)t\right]\cos\left[(2\omega_0 + \varepsilon)\,t\right] = \frac{1}{2}\cos\underbrace{\left[3\left(\omega_0 + \frac{1}{2}\varepsilon\right)t\right]}_{\text{higher freq term}} + \frac{1}{2}\cos\left[\left(\omega_0 + \frac{1}{2}\varepsilon\right)t\right].$$

The term with frequency $3(\omega_0 + \frac{1}{2}\varepsilon)$ may be shown to be higher order with respect to h. Such higher frequency terms create small amplitude perturbations of the solution and are neglected. (You shall see on a problem set how such terms arise from nonlinearities.)

Also, we shall retain only those terms that are first-order in ε. Thus we neglect the $\mathcal{O}(\varepsilon^2)$ associated with accelerations $\ddot{\theta}$. We also assume

$$\dot{a} \sim \varepsilon a, \qquad \dot{b} \sim \varepsilon b,$$

and thereby neglect $\ddot{a}$, $\ddot{b}$. These assumptions are validated by the final result.

*That is, we use $\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$ and $\sin A\sin B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$.

The result of the substitution is

$$
\begin{aligned}
- &\left(2\dot{a} + b\varepsilon + \frac{1}{2}h\omega_0 b\right)\omega_0 \sin\left[\left(\omega_0 + \frac{1}{2}\varepsilon\right)t\right] \\
+ &\left(2\dot{b} - a\varepsilon + \frac{1}{2}h\omega_0 a\right)\omega_0 \cos\left[\left(\omega_0 + \frac{1}{2}\varepsilon\right)t\right] = 0
\end{aligned}
$$

For this expression to be true for all t, the coefficients of both cos and sin must equal zero, i.e,

$$
\begin{aligned}
2\dot{a} + b\varepsilon + \frac{1}{2}h\omega_0 b &= 0 \\
2\dot{b} - a\varepsilon + \frac{1}{2}h\omega_0 a &= 0
\end{aligned}
$$

We seek solutions $a(t) \propto e^{\mu t}$ and $b(t) \propto e^{\mu t}$.
Thus, substitute $\dot{a} = \mu a$, $\dot{b} = \mu b$:

$$
\begin{aligned}
\mu a + \frac{1}{2}b\left(\varepsilon + \frac{1}{2}h\omega_0\right) &= 0 \\
\frac{1}{2}a\left(\varepsilon - \frac{1}{2}h\omega_0\right) - \mu b &= 0
\end{aligned}
$$

These equations have a solution when the determinant of the coefficients of a and b vanishes. Thus

$$
-\mu^2 - \frac{1}{4}\left[\varepsilon^2 - \left(\frac{1}{2}h\omega_0\right)^2\right] = 0
$$

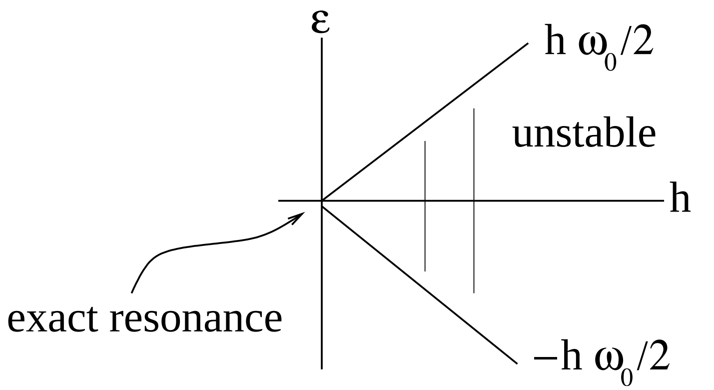
or

$$
\text{or } \mu^2 = \frac{1}{4}\left[\left(\frac{1}{2}h\omega_0\right)^2 - \varepsilon^2\right] \tag{18}
$$

Instability (parametric resonance) occurs when μ is real and positive. This will occur when $\mu^2 > 0$, or

$$
-\frac{1}{2}h\omega_0 \ < \ \varepsilon \ < \ \frac{1}{2}h\omega_0.
$$

This result is summarized in the following *phase diagram*:



Thus we see that resonance occurs not only for $\varepsilon = 0$, but for a range of ε within the lines $\pm h\omega_0/2$.

The larger the forcing amplitude, the less necessary it is for the force to be exactly resonant ($\omega = 0$).

Conversely, infinitesimal forcing (i.e., h) is sufficient for instability so long as $\varepsilon \to 0$.

6.4 Damping

The damped parametric pendulum is

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega_0^2 \left[1 + h\cos((2\omega_0 + \varepsilon)t)\right]\theta = 0$$

For unforced oscillations ($h = 0$), damping produces solutions like

$$\theta(t) \simeq e^{-\gamma t} \times \text{(oscillation)}.$$

For $h > 0$, we expect

$$\theta(t) \sim e^{(\mu-\gamma)t} \times \text{(oscillation)}$$

where the factor $e^{\mu t}$ results from the periodic forcing.

The instability boundary is therefore no longer given by $\mu = 0$ (i.e., equation (18)), but by

$$\mu - \gamma = 0.$$

Instability thus occurs for $\mu^2 > \gamma^2$. Using equation (18), we obtain

$$\mu^2 = \frac{1}{4}\left[\left(\frac{1}{2}h\omega_0\right)^2 - \varepsilon^2\right] > \gamma^2$$

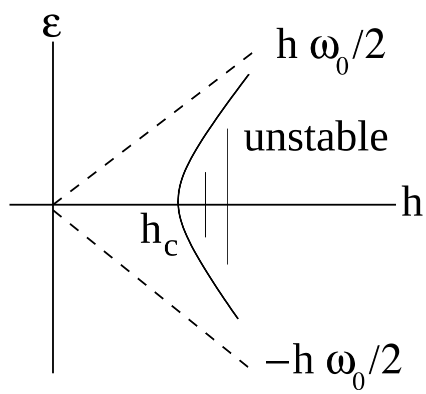
or

$$-\left[\left(\frac{1}{2}h\omega_0\right)^2 - 4\gamma^2\right]^{1/2} \ < \ \varepsilon \ < \ \left[\left(\frac{1}{2}h\omega_0\right)^2 - 4\gamma^2\right]^{1/2}$$

Setting $\varepsilon = 0$ (exact resonance) shows that instability is only possible when the quantity in the brackets is positive. Thus we require $h > h_c$, where

$$h_c = \frac{4\gamma}{\omega_0}$$

The phase diagram is therefore modified accordingly:



6.5 Further physical insight

We have seen that the instability is strongest at exact resonance.

That is, if the natural period of the pendulum is $T_{\rm nat}$, the most unstable excitation has period

$$T_{\rm exc} = \frac{1}{2}T_{\rm nat}$$

corresponding to angular frequencies

$$\omega_{\rm exc} = 2\omega_0$$

This is called a *subharmonic* instability, because instability results at half the frequency of the excitation (i.e., $\omega_0 = \omega_{\rm exc}/2$).

What other periods of excitation would be expected to lead to instability for small h?

Any integer multiple of T_{nat} suffices. Why? Because this would correspond roughly to, say, pushing a child on a swing every nth time he/she arrives at maximum height.

Therefore (weaker) instabilities will occur for

$$T_{\text{exc}} = \text{integer} \times \left(\frac{1}{2}T_{\text{nat}}\right).$$