12 Hénon attractor

The chaotic phenomena of the Lorenz equations may be exhibited by even simpler systems.

We now consider a discrete-time, 2-D mapping of the plane into itself. The points in $\mathbb{R}^2$ are considered to be the the Poincaré section of a flow in higher dimensions, say, $\mathbb{R}^3$.

The restriction that $d > 2$ for a strange attractor does not apply, because maps generate discrete points; thus the flow is not restricted by continuity (i.e., lines of points need not be parallel).

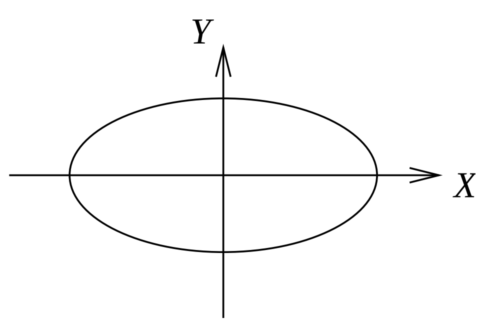
12.1 The Hénon map

The discrete time, 2-D mapping of Hénon is

$$
\begin{aligned}
X_{k+1} &= Y_k + 1 - \alpha X_k^2 \\
Y_{k+1} &= \beta X_k
\end{aligned}
$$

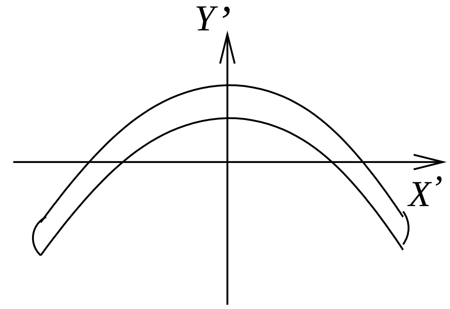
- α controls the nonlinearity.
- β controls the dissipation.

Pictorially, we may consider a set of initial conditions given by an ellipse:



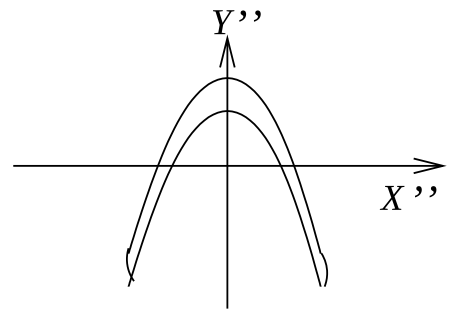
Now bend the elllipse, but preserve the area inside it (we shall soon quantify area preservation):

Map T_1 :
$$
\begin{aligned}
X' &= X \\
Y' &= 1 - \alpha X^2 + Y
\end{aligned}
$$



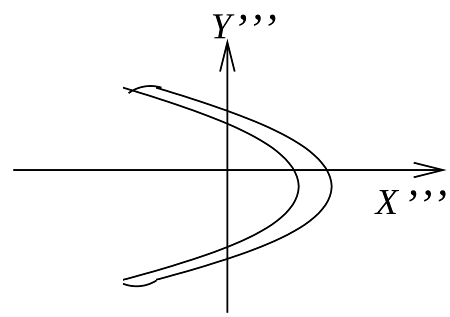
Next, contract in the x-direction ($|\beta| < 1$)

Map T_2 :
$$
\begin{aligned}
X'' &= \beta X' \\
Y'' &= Y'
\end{aligned}
$$



Finally, reorient along the x axis (i.e. flip across the diagonal).

Map T_3 :
$$
\begin{aligned}
X''' &= Y'' \\
Y''' &= X''
\end{aligned}
$$



The composite of these maps is

$$T = T_3 \circ T_2 \circ T_1.$$

We readily find that T is the Hénon map:

$$
\begin{aligned}
X''' &= 1 - \alpha X^2 + Y \\
Y''' &= \beta X
\end{aligned}
$$

12.2 Dissipation

The rate of dissipation may be quantified directly from the mapping via the Jacobian.

We write the map as

$$\begin{aligned} X_{k+1} &= f(X_k, Y_k) \\ Y_{k+1} &= g(X_k, Y_k) \end{aligned}$$

Infinitesimal changes in mapped quantities as a function of infinitesimal changes in inputs follow

$$\mathrm{d}f = \frac{\partial f}{\partial X_k}\mathrm{d}X_k + \frac{\partial f}{\partial Y_k}\mathrm{d}Y_k$$

We may approximate, to first order, the increment ΔX_{k+1} due to small increments $(\Delta X_k, \Delta Y_k)$ as

$$\Delta X_{k+1} \simeq \frac{\partial f}{\partial X_k}\Delta X_k + \frac{\partial f}{\partial Y_k}\Delta Y_k$$

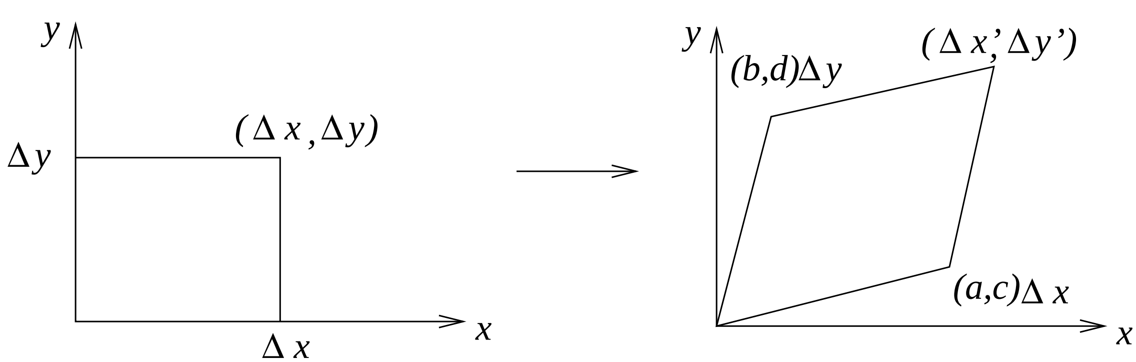
When $(\Delta X_k, \Delta Y_k)$ are perturbations about a point (x_0, y_0), we have, to first order,

$$\begin{bmatrix} \Delta X_{k+1} \\ \Delta Y_{k+1} \end{bmatrix} = \begin{bmatrix} f'_{X_k}(x_0, y_0) & f'_{Y_k}(x_0, y_0) \\ g'_{X_k}(x_0, y_0) & g'_{Y_k}(x_0, y_0) \end{bmatrix} \begin{bmatrix} \Delta X_k \\ \Delta Y_k \end{bmatrix}.$$

Rewrite simply as

$$\begin{bmatrix} \Delta x' \\ \Delta y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}.$$

Geometrically, this system describes the transformation of a rectangular area determined by the vertex $(\Delta x, \Delta y)$ to a parallelogram as follows:



Here we have taken account of transformations like

$$\begin{aligned}(\Delta x, 0) &\rightarrow (a\Delta x, c\Delta x)\\ (0, \Delta y) &\rightarrow (b\Delta y, d\Delta y)\end{aligned}$$

If the original rectangle has unit area (i.e., $\Delta x \Delta y = 1$), then the area of the parallelogram is given by the magnitude of the cross product of (a, c) and (b, d), or, in general, the Jacobian determinant

$$J = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} \dfrac{\partial X_{k+1}}{\partial X_k} & \dfrac{\partial X_{k+1}}{\partial Y_k} \\ & \\ \dfrac{\partial Y_{k+1}}{\partial X_k} & \dfrac{\partial Y_{k+1}}{\partial Y_k} \end{vmatrix}_{(x_0, y_0)}$$

Therefore

$$\begin{aligned}|J| > 1 &\implies \text{dilation}\\ |J| < 1 &\implies \text{contraction}\end{aligned}$$

For the Hénon map,

$$J = \begin{vmatrix} -2\alpha X_k & 1 \\ \beta & 0 \end{vmatrix} = -\beta$$

Thus areas are multipled at each iteration by $|\beta|$.

After k iterations of the map, an initial area a_0 becomes

$$a_k = a_0 |\beta|^k.$$

12.3 Numerical simulations

Hénon chose $\alpha = 1.4$, $\beta = 0.3$. The dissipation is thus considerably less than the factor of 10^{-6} in the Lorenz model.

An illustration of the attractor is given by

$\boxed{\text{BPV, Figure VI.19}}$

Numerical simulations show the basin of attraction to be quite complex.

Sensitivity to initial conditions is confirmed by

$\boxed{\text{BPV, Figure VI.20}}$

The weak dissipation allows one to see the fractal structure induced by the repetitive folding

$\boxed{\text{BPV, Figure VI.21}}$

Note the apparent scale-invariance: at each magnification of scale, we see that the upper line is composed of 3 separate lines.

The fractal dimension $D = 1.26$. (We shall soon discuss how this is computed.)

The action of the Hénon map *near* the attractor is evident in the deformation of a small circle of initial conditions:

$\boxed{\text{BPV, Figure VI.22}}$

At the scale of the attractor we can see the combined effects of *stretching* and *folding*:

$\boxed{\text{BPV, Figure VI.23}}$