# Problem Set 4

**Note: Problem 4 includes lab work—see footnote at the bottom of Page 3.**

Suggested reading: Strogatz, pp. 215–217, Baker and Gollub, pp. 27–35.

1. This problem illustrates one way in which perturbation methods may be applied to the nonlinear forced pendulum equation.

   The equation for undamped forced oscillations of a pendulum is

   $$\ddot{\theta} + \omega_0^2 \sin\theta = F\cos\omega t \tag{1}$$

   where $\omega_0 > 0$ is the natural frequency and $F\cos\omega t$ is the forcing term. For moderate swings, $\sin\theta \approx \theta - \theta^3/6$. Substituting this into Eq. 1 gives the nonlinear equation

   $$\ddot{\theta} + \omega_0^2\left(\theta - \frac{\theta^3}{6}\right) = F\cos\omega t. \tag{2}$$

   a) *Non-dimensionalization.* Show that equation 2 may be rewritten as

   $$\ddot{\theta} + \Omega^2\theta - \frac{1}{6}\Omega^2\theta^3 = \Gamma\cos\tau \tag{3}$$

   where we have used the dimensionless variables $\tau = \omega t$, $\Omega^2 = \omega_0^2/\omega^2$, and $\Gamma = F/\omega^2$, and where the derivatives are now with respect to $\tau$.

   b) *Series expansion in powers of a small parameter.* We consider the family of equations

   $$\ddot{\theta} + \Omega^2\theta - \epsilon\theta^3 = \Gamma\cos\tau \tag{4}$$

   where $\epsilon \geq 0$ is a small parameter ($\ll 1$). We note that equation 3 corresponds to the particular case $\epsilon = \Omega^2/6$. Note that as $\epsilon \to 0$, we obtain the linearized equation

   $$\ddot{\theta} + \Omega^2\theta = \Gamma\cos\tau \tag{5}$$

   which can be solved exactly. The simplest perturbation method represents the solution of Eq. 4 as a series in powers of $\epsilon$:

   $$\theta(\epsilon,\tau) = \theta_0(\tau) + \epsilon\theta_1(\tau) + \epsilon^2\theta_2(\tau) + \cdots \tag{6}$$

   where $\theta(0,\tau) = \theta_0(\tau)$ is the solution to Eq. 5 ($\epsilon = 0$). Substitute Eq. 6 into Eq. 4, retaining terms only up to order $\epsilon^2$. By matching corresponding orders of $\epsilon$ (i.e., the coefficients of powers of $\epsilon$ must balance), obtain the following system of ODE's for $\theta_0$, $\theta_1$, and $\theta_2$:

   $$\begin{aligned}
   \ddot{\theta}_0 + \Omega^2\theta_0 &= \Gamma\cos\tau, & (7)\\
   \ddot{\theta}_1 + \Omega^2\theta_1 &= \theta_0^3, & (8)\\
   \ddot{\theta}_2 + \Omega^2\theta_2 &= 3\theta_0^2\theta_1. & (9)
   \end{aligned}$$

c) *The first-order solution.* We can proceed to solve $\{\theta_i\}$ order by order, starting from $\theta_0$. Consider only periodic solutions having period $2\pi$, the period of the forcing term. In symbols, this means that we consider only solutions satisfying $\theta(\epsilon, t + 2\pi) = \theta(\epsilon, t)$. Furthermore, we assume that $\Omega \neq$ odd integer.

(i) Solve Eq. 7 and show that the only solution with period $2\pi$ and $\Omega$ not an integer is

$$\theta_0(\tau) = \frac{\Gamma}{\Omega^2 - 1} \cos \tau. \tag{10}$$

(ii) Obtain $\theta_1(\tau)$ by solving Eq. 8 after substitution of the solution for $\theta_0$ obtained above. Keep only solutions with period $2\pi$. (Hint: use the trigonometric identity $\cos^3 x = \frac{3}{4} \cos x + \frac{1}{4} \cos 3x$.)

(iii) Using your solutions for $\theta_0$ and $\theta_1$, write the first order estimate of the solution to equation 2.

2. Determine analytically the discrete Fourier transform,

$$\hat{x}_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j \exp\left(-i \frac{2\pi j k}{N}\right),$$

for each of the following series. Give both the real and imaginary parts. Also determine the power spectrum, $|\hat{x}_k|^2$, for each series.

a) $x_j = \cos(2\pi m j/N), \quad j \in \{0, 1, \ldots, N-1\}, \quad m$ an integer.

b) $x_j = \sin(2\pi m j/N), \quad j \in \{0, 1, \ldots, N-1\}, \quad m$ an integer.

c) $x_j = \cos(2\pi m_1 j/N) \cos(2\pi m_2 j/N), \quad j \in \{0, 1, \ldots N-1\}, \quad m_1, m_2$ integers.

3. a) Confirm your answers to Question 2 numerically using Matlab. Download the files `makesignal.m` and `fourier.m` and read the comments in the files.

Modify `makesignal.m` to generate the required data. The Fourier transform program uses a method called the *fast Fourier transform*, which requires that the number of data be a power of 2, so use $N = 32$. You should try several values of $m, m_1$ and $m_2$, but hand in only power spectra for the cases $m = 8, m_1 = 8$ and $m_2 = 6$.

b) For the series in question 2a), compare the power spectrum when $m = 8$ to the spectrum when $m = 8.4$. What differences in the two spectra can you explain theoretically?

4. Obtain data[2] in the form of time series $\theta(t)$ from our real driven pendulum for at least three different combinations of driving amplitude and frequency (and thus three different dynamical states). Using each of these time series, perform the following analysis (please hand in graphs to support your conclusions):

   a) Compute the power spectrum for each time series.

   b) Are your time series *periodic* (i.e., with just one frequency), *quasi-periodic* (i.e., periodic but the sum of more than one frequency), or *aperiodic* (i.e., the signal never repeats itself)? If they are periodic or quasiperiodic, what is the fundamental frequency (i.e., the lowest frequency of oscillation)? Justify your answer with both the time series and the power spectra. Explain the presence of any peak other than the fundamental frequency that you find in your spectra. You may wish to view the logarithm of your spectra to view all the peaks.

   c) For each time series, calculate the *autocorrelation function*,

   $$\psi_m = \frac{1}{N} \sum_{j=0}^{N-1} x_j x_{j+m}$$

   of the time series, using either the Wiener-Khintchin theorem or explicit calculation of the above summation. Can you relate qualitatively the principal features of the autocorrelation to both the time series and the power spectrum? Are the pendulum's fluctuations predictable for long times? How long?

[2] You will need to visit the lab, where the TAs will assist you. (You may come in groups if you like.)