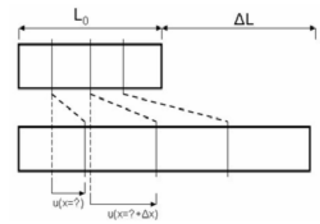
Stress/Strain

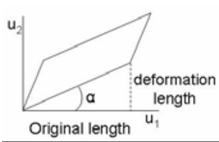
Stress = $\text{Force}/\text{Cross-sectional area}$ = $\sigma_{face-direction} = \sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$ normal stresses (applied+hydrostatic forces) on diagonal, symmetric

<u>Simple shear</u>: fixed bottom. $\sigma_{12} = \sigma_{21} \neq 0$ <u>biaxial tension</u> $\sigma_{11}, \sigma_{22} \neq 0$

Strain = $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ symmetric tensor



uniform strain = nonuniform displacement!
Uniform displacement = translation
Normal strain: Δ lengths; Shear strain: Δ angles



shear strain = $\frac{\text{deformation}}{\text{original length}}\ \alpha \approx \tan\alpha = \frac{\partial u}{\partial x}$

-gradient of u1 in the x2 direction

Linear Elasticity: material response is time independent

Assumptions: homogeneous, isotropic, linear (stress is linear function of strain), elastic (stress-strain are directly related, no time component).

Incompressible? $\upsilon = 0.5 \xrightarrow{\text{uniaxial tension}} -\varepsilon_{transverse}/\varepsilon_{axial}$ $\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 0 = \Delta\text{Vol}/\text{Vol}$

[1] Equilibrium (σ) [2] Compatibility (ε) [3] Constitutive Laws (σ-ε)

$\sigma_{ij} = \Sigma\Sigma C_{ijkl}\varepsilon_{ij}$ C→ 81 constants $\xrightarrow{\text{symmetry}} 36 \xrightarrow{\text{isotropy}} 2$

$\sigma_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$ OR $\varepsilon_{ij} = \frac{1+\upsilon}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$ where $\lambda = \frac{2G\nu}{1-2\nu} = \frac{E\nu}{(1+\nu)(1-2\nu)}$

Confined compression? walls apply stresses and strains at walls are 0.

Viscoelasticity: material response is time-dependent

Models:

General Creep: step σ → jump+inc'ing ε (ε_∞ →constant, solid. ε_∞ →∞, fluid.)

Creep Compliance: J(t) = $\varepsilon(t)/\sigma_0$. $\varepsilon(t) = \int_0^t J(t-\tau)\frac{\partial\sigma(\tau)}{\partial\tau}d\tau$

General σ relax.: step ε → jump+dec'ing σ (σ_∞ →constant, solid. σ_∞ →0, fluid.)

Relaxation modulus: G(t) = $\sigma(t)/\varepsilon_0$. $\sigma(t) = \int_0^t G(t-\tau)\frac{\partial\varepsilon(\tau)}{\partial\tau}d\tau$

[1] Equilibrium: σ's = for elements in series, add for elements in parallel
[2] Compatibility: ε's = for elements in parallel, add for elements in series
[3] Constitutive: Spring: $\varepsilon = \sigma/E$; $\dot{\varepsilon} = \dot{\sigma}/E$ Dashpot: $\dot{\varepsilon} = \sigma/\eta$ ($\eta = \text{Pa}\cdot\text{s}$)

Standard Models: (J(t), G(t), differential equations)

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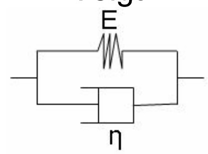
<td>
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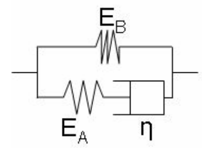
<td>
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Oscillatory testing: time lag between applied stress and reactive strain
Storage Modulus (~elastic, G'=Re(G*)), Loss Modulus (~viscous, G''=Im(G*))
$\varepsilon(t) = \varepsilon_0 \sin \omega t$; $\sigma(t) = \sigma_0 \sin(\omega t + \delta)$ → δ =0 for solid, $\pi/2$ for fluid

Biological Basis for Mech. Properties: collagen (~steel cables, many types!), elastin (~rubber bands), proteoglycan+GAGs (~charged sponge)

Poroelasticity: time dependent response of fluid-saturated, compressible solid

- pressure acts on *both* the fluid and the solid

(1) Constitutive relation (3D) $\sigma_{ij} = 2G\varepsilon_{ij} + \lambda \varepsilon_{kk}\delta_{ij} - p\delta_{ij}$

(2) Conservation of mass (1D)

mean flow vel. rel. to solid avg'd over area= $U_1 = \frac{-\partial u_1}{\partial t} + U_0$

(3) Conservation of momentum/Force Balance (1D) $\frac{\partial \sigma_{11}}{\partial x_1} = 0$

(4) Darcy's law (1D) $U_1 = -k\frac{\partial P}{\partial x_1}$, k = hydraulic permeability

(5) Definition of strain $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

1D Poroelasticity (confined compression): $\frac{\partial u_1}{\partial t} = Hk\frac{\partial^2 u_1}{\partial x_1^2}$; $\tau_{char} \approx Hk/L^2$; H=2G+ λ

Porosity = $\phi = \frac{Vol_{fluid}}{Vol_{solid} + Vol_{fluid}}$

Reduced Forms: [1] U_0 =0, impermeable bottom, and $\partial u_1/\partial t = 0$ → $\partial u_1/\partial x_1$ =constant

[2] $\partial u_1/\partial t = 0$, $U_0 \neq 0$ → $\partial^2 u/\partial x_1$ = $-U_0/Hk$ [3] U_0 =0, $\partial u_1/\partial t \neq 0$ → $\frac{\partial u_1}{\partial t} = Hk\frac{\partial^2 u_1}{\partial x_1^2}$