Massachusetts Institute of Technology
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2.002 Mechanics and Materials II
Spring 2004

Laboratory Module No. 1
Elastic behavior in tension, bending, buckling, and vibration.

1 Objectives

In this laboratory session we will review elementary concepts concerning the isotropic linear elastic behavior of materials and structures. We will consider elastic loading in simple tension, cantilever beam-bending, vibration and buckling. We also introduce experimental methods for quantifying elastic properties of materials and elastic response of structural components.

2 Lab Tasks

In this laboratory module we will perform the following tasks:

- review concepts of stress and strain, and the definition of the isotropic elastic constants: Young's modulus (E) and Poisson's ratio, (ν);
- conduct an elastic-level tension test on a strip specimen of 6061-T6 aluminum alloy, mounted with two strain gages. The load and applied to the specimen will be monitored (load cell), and strain gauges mounted both parallel and perpendicular the bar axis will monitor the respective strain components (axial and transverse).
- review elastic analysis of tip-loaded cantilever beams and use a cantilever beam as a leaf spring, inferring E-values from both local bending strain measurements and from tip deflection.
- review elastic theory for lateral buckling of slender axially-compressed members (Euler buckling), and conduct simple buckling experiments to obtain estimates of E in slender members of different materials, cross-sections, and lengths.
- review elastic beam vibration, and use the natural vibration frequency of cantilever beams to estimate E of the material.

3 Lab Assignments: Specific Questions to Answer

1. Using the dimensions of the tensile specimen, recorded values of the axial load, and the axial and transverse strain gage readings, determine Young's modulus (E) and the Poisson ratio (ν) for this material.

2. Record the specimen dimensions and strain gage locations on the instrumented cantilever. Record the values for axial strain at each gage, and the lateral tip deflection, for the tip loadings applied in class. How well do these measurements agree with theoretical values based on beam theory? How well do they predict E? Discuss.

3. Use the elastic structural stiffnesses (load/tip-displacement) measured via cantilever bending of the specimens, along with specimen dimensions, to infer an E-value for each material. How well do these values agree with other measurements of E? Discuss.

4. Use the critical elastic buckling load measured on the specimens you are given to estimate E for that material. How well do these values agree with other measurements of E?

5. Use the natural frequencies of the vibrating cantilever beams measured in the lab, along with the specimen dimensions and the appropriate mass values to estimate Young's modulus, E.

6. Do the different test methods (tension test with strain gauges; instrumented cantilever and cantilever stiffness; natural frequency, buckling) provide consistent estimates of E? Discuss.

4 Review of Cantilever Beam Bending

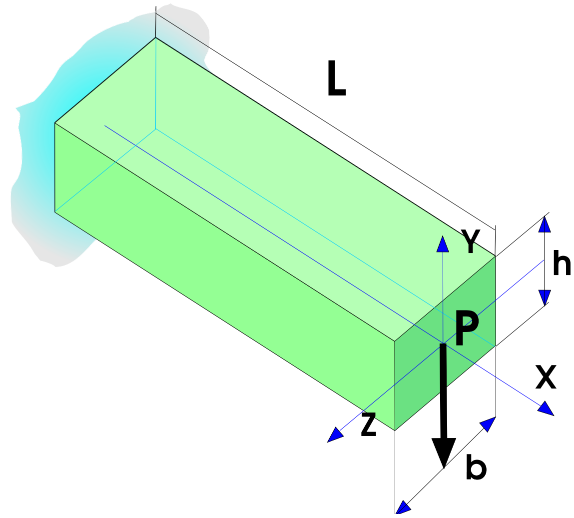
(See also Crandall, Dahl and Lardner, sections 3.5; 7.5; 8.3)

Shear force and bending moment equations:

The distributed loading (force/length) is $q(x)$; transverse shear force is $V(x)$; and bending moment is $M(x)$. For the cantilever, $q(x) = 0$ in $0 < x < L$. (See the CDL text for sign conventions on positive deflection, shear force, distributed load, and bending moment.)

$$\frac{dV(x)}{dx} + q(x) = 0; \quad \frac{dV(x)}{dx} = 0 \quad \Rightarrow \quad V(x) = \text{constant} = -P; \tag{4.1}$$

$$\frac{dM(x)}{dx} + V(x) = 0; \frac{dM(x)}{dx} = -V(x) = +P; \quad \Rightarrow \quad M(x) = Px + \text{constant} = -P(L-x). \tag{4.2}$$



Here the integration constants can be directly evaluated from free body diagrams of the end-portion of the cantilever in the interval $[x, L]$.

Axial stress at a generic point (x,y,z):

$$\sigma_{xx}(x, y, z) = -\frac{M(x)y}{I} = \frac{P(L-x)y}{I} \tag{4.3}$$

Note: in writing the previous expressions for $-M(x) = P(L-x)$, we have assumed that the origin of the x-axis is at the base of the cantilever ($x = 0$), and that the tip where load is applied is at $x = L$; thus, the generic coordinate value "x" measures distance from the base of the cantilever. The drawing of the coordinate axes in the figure can give rise to confusion here.

Axial strain at a generic point (x,y,z):

$$\epsilon_{xx}(x, y, z) = \frac{\sigma_{xx}(x, y, z)}{E} = -\frac{M(x)y}{EI} = \frac{P(L-x)y}{EI}, \tag{4.4}$$

where $I \equiv \int y^2\, dA$ is the area moment of inertia of the cross-section, which, for rectangular cross-sections of this orientation, is equal to:

$$I = \frac{bh^3}{12}, \tag{4.5}$$

with b the width and h the thickness of the beam.

Assuming that axial strain $\epsilon_{(\mathrm{surf})}(x)$ at position x has been measured on the surface of the beam (at $|y| = h/2$), in the presence of cantilever load P, we can obtain a local-strain-based estimate of the elastic modulus E as

$$E \doteq E_{(\mathrm{gaged\ cantilever})} = \frac{|M(x)|\, h/2}{I\, |\epsilon_{(\mathrm{surf})}(x)|} = \frac{6P(L-x)}{bh^2\, |\epsilon_{(\mathrm{surf})}(x)|}. \tag{4.6}$$

Here we use absolute value of the bending moment and the surface strain (even if it is measured on the compression side) in order to get a positive modulus. Convince yourself that this is mathematically correct in either case, $y = \pm h/2$.

An axially-oriented strain gage, mounted on the face of a cantilever beam, gives a signal proportional to the bending moment at its location; from the differential equation of moment equilibrium, the difference in signal ($\propto \Delta M$) between two such gages, mounted a distance Δx apart, provides a direct measure of shear force ($V = -\Delta M/\Delta x$).

Lateral displacement $v(x)$ and rotation $\phi(x)$:

Within the assumptions of traditional elastic beam theory, the lateral displacement of the beam in the vertical ($+y$) direction is $v(x)$, the [small] counter-clockwise rotation about the $+z$-axis is $\phi(x) \doteq v'(x)$, and the curvature is $\kappa(x) \doteq \phi'(x) \doteq v''(x)$. For elastic response, the curvature is related to the bending moment by

$$\frac{d^2v(x)}{dx^2} = \frac{d\phi(x)}{dx} = \kappa(x) = \frac{M(x)}{EI} = \frac{-P(L-x)}{EI}. \tag{4.7}$$

This equation can be integrated once, introducing a constant of integration C_1 (to be determined):

$$\frac{dv(x)}{dx} = \phi(x) = \frac{-P}{EI}\left(Lx - \frac{x^2}{2} + C_1\right). \tag{4.8}$$

Using the condition that there is no rotation at the base ($\phi(x=0) = 0$) provides the numerical value of the integration constant as $C_1 = 0$. A second integration, integration constant, and the boundary condition $v(x=0) = 0$ provide the complete displacement field as

$$v(x) = -\frac{Px^2}{6EI}(3L - x). \tag{4.9}$$

The lateral displacement at the tip, $x = L$, is termed "δ"; its evaluation [positive 'downward', assuming tip load in the $-y$-direction] provides

$$\delta \equiv -v(x = L) = +\frac{PL^3}{3EI}; \tag{4.10}$$

the stiffness of the cantilever can be defined as

$$k_{(\text{cantilever})} \equiv \frac{P}{\delta} = \frac{3EI}{L^3}. \tag{4.11}$$

A re-arrangement of this equation provides a stiffness-based estimate of the elastic modulus as

$$E \doteq E_{(\text{cantilever stiffness})} \equiv \frac{PL^3}{3I\delta}. \tag{4.12}$$

5 Elastic Buckling of Long Slender Columns

(See also Crandall, Dahl, and Lardner, section 9.4)

The elastic buckling load of an axially compressed cylindrical body (a column) can be expressed as

$$F_{cr} = c \times \frac{\pi^2 EI}{L^2}, \tag{5.1}$$

where L is the length of the column, I is the *minimum* centroidal area moment of inertia of the section, E is the Young's modulus, and c is a dimensionless constant which depends on the fixity of the end conditions. For axial compression load F which are smaller than F_{cr}, the column remains substantially straight, but as $F \to F_{cr}$, the critical buckling load, equilibrium of a straight column is not stable, and buckling to a laterally deflected shape occurs. The shape of the deflected column is called the buckling mode, and the particular buckling mode, along with its critical buckling load, depends on the boundary conditions applied at the column ends.

Equivalently, the buckling load can be predicted from the following formula:

$$F_{cr} = \frac{\pi^2 EI}{(L')^2}, \tag{5.2}$$

where the effective length, L', is related to the actual length, L, by

$$L' = KL \tag{5.3}$$

with

$$K = \frac{1}{\sqrt{c}}. \tag{5.4}$$

A table of the *effective length factors*, K, is given below, along with schematic illustrations of the buckling modes.

Under simple "pinned" end supports (lateral displacement restrained; free rotation implies zero moment at ends), $K = 1$, so the critical load at which non-zero lateral buckling in the central portion is observed provides a buckling-force-based estimate of the elastic Young's modulus as

$$E \doteq E_{(\text{buckling})} \equiv \frac{F_{cr} L^2}{\pi^2 I}. \tag{5.5}$$

Note that, for solid circular cross-sections of diameter "d", the area moment of inertia is $I = \pi d^4/64$.

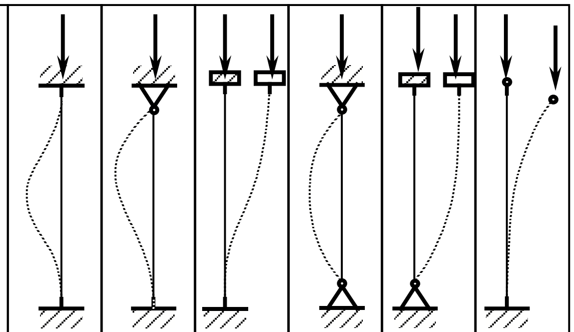
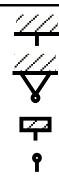
Buckled shape of column is shown by dotted line						
Theoretical K value	0.5	0.7	1.0	1.0	2.0	2.0
Recommended K (design value when ideal end conditions are approximated)	0.65	0.8	1.2	1.0	2.1	2.0
End condition code		Rotation fixed and translation fixed Rotation free and translation fixed Rotation fixed and translation free Rotation free and translation free				

Figure 1: Effective length factors and buckling mode shapes.

6 Review of Isotropic Linear Elasticity

(See also Crandall, Dahl, and Lardner, section 5.4)

The theory of isotropic linear elasticity is the most common constitutive relation used to describe the mechanical behavior of engineering solids. Its purpose is to quantify relations between the components (σ_{ij}) of the stress tensor, $\boldsymbol{\sigma}$, and those (ϵ_{ij}) of the strain tensor, $\boldsymbol{\epsilon}$. Here, for shorthand, we use the cartesian subscript notation for the matrix of stress (or strain) components, with the understanding that "i" and "j" can each assume values from one to three, indicating, in turn, three orthogonal spatial directions. In particular, we can connect with the alternate "$x-y-z$" notation by equating direction "1" with "x"; direction "2" with "y"; and direction "3" with "z". Thus, the tensile stress in the x (or 1) direction can be expressed as "σ_{xx}" or as "σ_{11}". Shear stress components are the off-diagonal components of the stress tensor matrix; thus "σ_{xy}" is also "σ_{12}". [1]

When the macroscopically-measured properties of the material under consideration are independent of the particular choice of the three orthogonal directions used to describe the stress and strain components, the material is said to be "isotropic". In an isotropic linear elastic material, only two independent material properties are

[1] We note that, in some texts, a different symbol (e.g., "τ") is used to denote shear stress components, while the symbol "σ" is used for normal stress components. Such a two-symbol notational convention is not needed in the double-subscript notation, since equality of subscripts (e.g., σ_{22} ($= \sigma_{yy}$)) intrinsically denotes a normal stress, while unequal subscripts (e.g., σ_{23}) intrinsically denote a shear stress component.

required to quantify the constitutive relation between stress and strain components. One common choice for the two independent material properties consists of the pair E and ν, the Young's (or tensile) modulus and Poisson ratio, respectively. Written in full, the constitutive relation is

$$\epsilon_{ij} = \frac{1}{E}\left[(1+\nu)\,\sigma_{ij} - \nu\delta_{ij}\left(\sum_{k=1}^{3}\sigma_{kk}\right)\right]. \tag{6.1}$$

Here "δ_{ij}" is the notation for components of the identity matrix, so that its value, for any particular choice of i and j, is

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{6.2}$$

Thus, $\delta_{11} = \delta_{22} = \delta_{33} = 1$ while $\delta_{12} = \delta_{21} = \delta_{13} = \delta_{31} = \delta_{23} = \delta_{32} = 0$. Since (6.1) can be evaluated for any combination of subscripts i and j, it is in fact a "shorthand" expressing 9 ($= 3 \times 3$) separate strain components in terms of the stress components.

For a shear stress/strain pair such as '$2-3$', (6.1) provides $\epsilon_{23} = (1+\nu)\sigma_{23}/E$. A third elastic material property, the shear modulus G, can also be used to describe the stress-strain relations; a typical result is $\gamma_{23} \equiv 2\epsilon_{23} = \sigma_{23}/G$, where "$\gamma$" is the engineering shear strain. The value of the G is not independent of the values of E and ν in an isotropic linear elastic solid; in particular, they are evidently related by

$$2G\,(1+\nu) = E. \tag{6.3}$$

For example, (6.1) could also be written in terms of G and ν as

$$\epsilon_{ij} = \frac{1}{2G}\left[\sigma_{ij} - \frac{\nu}{(1+\nu)}\delta_{ij}\left(\sum_{k=1}^{3}\sigma_{kk}\right)\right]. \tag{6.4}$$

Both of the expressions (6.1, 6.4) for the stress-strain relations are in fact a set of nine ($= 3 \times 3$) equations as i and j independently assume each of the values from 1 through 3. The nine resulting equations are not independent, as the matrices of both stress and strain components are symmetric (e.g., $\sigma_{23} = \sigma_{32}$), so only six independent equations are represented.

In a uniaxial tension test, where 'P' is the load in loading direction, x_1, and specimen cross-sectional area is A, the only non-zero stress component is $\sigma_{11} = P/A$; $\sigma_{ij} = 0$ otherwise. When these stress values are inserted into the full stress-strain relations (6.1), the non-zero strains are

$$\epsilon_{11} = \frac{\sigma_{11}}{E} = \frac{P}{AE}; \tag{6.5a}$$

$$\epsilon_{22} = \epsilon_{33} = -\nu\epsilon_{11} = -\frac{\nu\sigma_{11}}{E} = -\frac{\nu P}{AE}. \tag{6.5b}$$

Rearrangement of these equations then provides the following, strain-gage-instrumented tension-test-based estimates of elastic constants as

$$E = E_{(\text{strain-gaged tension})} = \frac{\sigma_{11}}{\epsilon_{11}} = \frac{P}{A\epsilon_{11}}; \tag{6.6a}$$

$$\nu = -\frac{\epsilon_{22}}{\epsilon_{11}}. \tag{6.6b}$$

7 Lateral Vibration of Cantilever Beams

7.1 Harmonic oscillator and tip-weighted cantilever beam vibration

The natural frequency of a simple harmonic oscillator depends on both the stiffness of the restoring (elastic) member in the system and the mass which is being accelerated/decelerated. For a rigid mass m connected to a massless spring of linear stiffness k (dimensions: force/length), having one end grounded while the other is attached to the moving mass, the natural frequency is simply

$$\omega = \sqrt{\frac{k}{m}}. \tag{7.1}$$

Here is it understood that the time-based displacement of the mass is given, for example, by

$$u(t) = u_0 \sin \omega t, \tag{7.2}$$

where u_0 is an arbitrary (but "sufficiently small") magnitude of (peak) displacement.

For the time-dependent motion (7.2), the velocity of the mass point is $\dot{u}(t) = \omega u_0 \cos \omega t$ and its acceleration is $\ddot{u}(t) = -\omega^2 u_0 \cos \omega t = -\omega^2 u(t)$. When the mass has moved one end of the grounded spring by amount $u(t)$, the spring exerts a restoring force on the mass of magnitude $f(t) = -ku(t)$. Newton's law tells us that $f(t) = m\ddot{u}(t)$; substituting in terms of $u(t)$ for both sides of the equation, $(-k + m\omega^2)u_0 \sin \omega t \equiv 0$. Since this must hold for all times, t, and for $u_0 \neq 0$, we directly recover (7.1).

Because of the periodicity of the sine function, displacements and velocities are equal at time intervals separated by the natural period, τ, where

$$\omega\tau = 2\pi \tag{7.3}$$

or

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}. \tag{7.4}$$

In any finite interval of time, Δt, the number of complete vibration cycles experienced by the oscillator, N, is

$$N = \frac{\Delta t}{\tau}, \tag{7.5}$$

providing the experimental measurement of the natural period as

$$\tau_{\text{(experimental)}} = \frac{\Delta t}{N}. \tag{7.6}$$

In turn, for a known mass m and an experimentally-observed period τ, the system stiffness can be predicted to be

$$k = m \left(\frac{2\pi}{\tau_{\text{(experimental)}}} \right)^2. \tag{7.7}$$

Now consider the case when a point mass m is mounted at the tip of a cantilever beam of length L, thickness h, and breadth b, as before, and vibrating in the plane for which the relevant area moment of inertia is I. If the mass density of the beam itself is given by ρ, then the mass of the beam within the region $0 \leq x \leq L$ is just $m_{\text{(beam)}} = \rho b h L$. Providing the tip mass is much larger than the mass of the beam ($m \gg m_{\text{ (beam)}}$), the beam's mass can be ignored by comparison, and the system is, approximately, a harmonic oscillator with mass m and a spring constant given by the stiffness of the cantilever itself:

$$k = k_{\text{(cantilever)}} = \frac{3EI}{L^3} = \frac{P}{\delta}. \tag{7.8}$$

By combining equations (7.7) and (7.8), an experimental value of the elastic modulus can be obtained from the natural frequency measurements as

$$E \doteq E_{\text{(harmonic oscillator)}} = \frac{mL^3}{3I} \left(\frac{2\pi}{\tau_{\text{ (experimental)}}} \right)^2. \tag{7.9}$$

7.2 Continuous uniform cantilever beams

At the opposite extreme, suppose that no concentrated tip mass is present on a vibrating cantilever: the mass elements which are being accelerated perpendicular to the axis of the beam are simply those of the beam itself. It is clear, however, that different infinitesimal mass elements $dm_{\text{(beam)}}(x) = \rho b h \, dx = \rho A \, dx$, located at different positions x along the beam, experience different lateral displacements. But at each location, the period of vibration is the same, since the overall beam has one and the same fundamental frequency of vibration.

We can anticipate the functional form for the natural frequency of such a distributed system on dimensional grounds: the harmonic oscillator gives us the clue that ω should scale as (read: 'be proportional to') the square-root of a quotient of stiffness divided by mass. But (a) what mass?; (b) what stiffness?; and (c) what proportionality constant? Since there is only one frequency, after all, we can choose the mass and the stiffness arbitrarily, leaving it up to a dimensionless constant α to "fix things up".

Returning to our vibrating cantilever beam, we have already evaluated "a" stiffness measure for the structure; namely the cantilever stiffness for concentrated tip loads:

$$k_{\text{(cantilever)}} = \frac{3EI}{L^3}. \tag{7.8}$$

Similarly, an obvious mass to associate with this vibration is the total mass of the beam in the free region $0 \leq x \leq L : m_{(\text{beam})} = \rho bhL$. Thus, guided by the form of the harmonic oscillator and pinning our hopes on a dimensionless factor "α" yet to be evaluated, we write the fundamental vibration frequency of a uniform cantilever beam as

$$\omega \doteq \omega_{(\text{lumped parameter})} = \sqrt{\frac{k_{(\text{cantilever})}}{\alpha m_{(\text{beam})}}} = \sqrt{\frac{3EI}{\alpha \rho bhL^4}} = \frac{\sqrt{3/\alpha}}{L^2}\sqrt{\frac{EI}{\rho A}}. \tag{7.10}$$

As presented here, the dimensionless factor α could perhaps be considered as a fraction accounting for the fact that not "all" of the vibrating beam mass is located at the tip, where the cantilever stiffness is evaluated. Alternatively, we might consider $1/\alpha$ as a "stiffness enhancement", reflecting the fact that the "effective stiffness" of the vibrating structure exceeds that of a tip-loaded cantilever of length L. For any reasonable choice of α, somewhere in the range $0 < \alpha < 1$, this simple "lumped parameter" model of the vibrating beam provides an estimate of the natural frequency.

But we know that a vibrating beam is, in fact, a continuous system, with a spectrum of natural frequencies and mode shapes, and such vibration problems can be solved by more advanced mathematical methods. An outline of the procedure is presented in the following section. For the vibrating uniform cantilever beam, it turns out that the lowest (first mode, or fundamental) frequency can be expressed as

$$\omega = \frac{\xi^{\star 2}}{L^2}\sqrt{\frac{EI}{\rho A}}, \tag{7.11}$$

where the numerical value of the dimensionless parameter $\xi^\star$ is obtained as the lowest positive root, $\xi = \xi^\star$, to the transcendental equation

$$1 + \cos\xi \cosh\xi = 0. \tag{7.12}$$

Obviously, the only solutions to (7.12) occur when $\cos\xi \, \cosh\xi = -1$; there are an infinite number of such ξ-values, but the smallest one occurs someplace just past the point where the cosine function first becomes negative, just beyond $\xi = \pi/2 \doteq 1.57$. Numerical (or graphical) solution gives the first root as $\xi = 1.875104 \equiv \xi^\star$. Evidently, by matching the frequency of this exact solution to our simple lumped parameter estimate of frequency, we see that the optimal (matching) value of α for the lowest mode is obtained by choosing

$$\alpha = \frac{3}{(\xi^\star)^4} = \frac{3}{(1.875194)^4} = 0.24267. \tag{7.13}$$

Combining all the equations of the vibrating uniform cantilever, the elastic modulus E of the beam material can be estimated in terms of the first-mode period (τ), beam geometric properties (L, A, and I) and material mass density (ρ) as

$$E \doteq E_{(\text{vibrating cantilever})} = \left(\frac{L}{\xi^\star}\right)^4 \left(\frac{\rho A}{I}\right) \left(\frac{2\pi}{\tau}\right)^2. \tag{7.14}$$

In making quantitative use of a result such as (7.14), it is vitally important to use consistent units; if available data is expressed in non-consistent or non-standard units, then appropriate conversions must be performed in order to correctly express the desired result in customary units (e. g., in GPa, for E).

8 Detailed Derivation of Cantilever Beam Vibration

This section contains a derivation of the fundamental mode shape and frequency for a vibrating cantilever beam. As such, it should be considered as background material for this lab. You are not expected to fully 'master' all of the material in this section. But you are likewise not expected to completely ignore it!

In Section 7.2 above, we rushed to equation (7.14), which could provide an estimate of E based on a uniform cantilever beam's vibration in its first natural mode. Most of the details were left out. Here, we quickly summarize those details, for completeness. It should be noted that vibration of beam structures is a well-developed area of study, and many references in structural dynamics contain fuller accounts.

We seek to find free vibration modes and frequencies for the cantilever beam described above. We will adopt classical Euler-Bernoulli beam dynamics theory to analyze the problem; this will provide adequate answers for our purposes. However, more refined beam theories, often termed "Timoshenko" beam theory, lead to slightly different results, especially for the higher frequencies (Timoshenko beam theory is generally more accurate for higher modes).

We look for solutions to the dynamics equations (Newton's laws!) in which the beam undergoes time- and space-dependent lateral displacement (vibration in the local y-direction) of the form:

$$v(x,t) = \bar{v}(x) \sin \omega t, \tag{8.1}$$

where ω is the natural frequency and $\bar{v}(x)$ is the associated mode shape of the vibration.

The overall approach consists of the following:

1. Develop equations of motion for an infinitesimal beam segment undergoing the postulated motion.

2. Use classical linear elastic beam theory to cast the driving "force" variables in terms of beam curvatures, etc., and on substituting into the equation of motion, obtain a differential equation, with frequency ω as a yet-undetermined parameter, which the mode shape function $\bar{v}(x)$ must satisfy.

3. Develop the general form for the solution of the resulting (4th-order) ordinary differential equation for $\bar{v}$, including 4 arbitrary constants of integration.

4. Use boundary conditions at the beam ends $x = 0$ and $x = L$ to evaluate the constants. We will find that a non-trivial solution is possible only if the frequency satisfies a particular transcendental "characteristic" equation; the spectrum of roots to this equation defines all possible natural frequencies ω_i, that the beam [model] can exhibit. There are an infinite number of such frequencies, as index $i = 1, 2, 3, \ldots,$. For each frequency ω_i, there is a unique mode "shape", $\bar{v}_i(x)$, but the amplitude of each mode shape is arbitrary (providing, of course, that it is sufficiently small to meet the small strain and small rotation limits of elastic beam theory).

5. For fixed beam geometry, the material property scaling the natural frequencies is the "specific stiffness", E/ρ; in fact, $\omega_i \propto \sqrt{E/\rho}$ for each mode.

(1.) In the absence of distributed loads ($q(x) \equiv 0$), the net y-direction force applied to an infinitesimal beam slice between "x" and "$x + dx$" is

$$\sum F_y = V(x + dx, t) - V(x, t) \simeq dx \left(\frac{\partial V(x, t)}{\partial x} \right). \tag{8.2}$$

Here $V(x, t)$ is the shear force in the beam at position x and time t.

The beam segment is instantaneously moving in the y-direction; its y-component of velocity is

$$\dot{v}(x, t) \equiv \frac{\partial v(x, t)}{\partial t} = \omega \bar{v}(x) \, \cos \omega t, \tag{8.3}$$

while its acceleration is

$$\ddot{v}(x, t) \equiv \frac{\partial^2 v(x, t)}{\partial t^2} = -\omega^2 \bar{v}(x) \, \sin \omega t. \tag{8.4}$$

The elemental mass of the slice is $dm = dx\,(\rho A)$, and application of Newton's laws provides, on cancelling the common non-zero factor dx,

$$\frac{\partial V(x, t)}{\partial x} + \omega^2 \rho A \bar{v}(x) \sin \omega t = 0. \tag{8.5}$$

The z-component of the moment balance equation for the slice provides, in similar fashion,

$$\frac{\partial M(x, t)}{\partial x} + V(x, t) = 0. \tag{8.6}$$

Equation (8.6) can be partially differentiated with respect to x, and on inserting the value of $\partial V/\partial x$ from (8.5), we obtain

$$\frac{\partial^2 M(x, t)}{\partial x^2} - \omega^2 \rho A \bar{v}(x) \, \sin \omega t = 0. \tag{8.7}$$

(2.) We can substitute the elastic beam curvature/bending moment equation $M(x, t) = EI\kappa(x, t)$, where $\kappa(x, t) = \partial^2 v(x, t)/\partial x^2$ is beam curvature, into (8.7), and perform the indicated partial derivatives to obtain:

$$\sin \omega t \, \left(\, EI \bar{v}''''(x) - \rho A \omega^2 \bar{v}(x) \, \right) = 0. \tag{8.8}$$

Since (8.8) must hold for all times 't', the factor in parentheses provides the differential equation for mode shape ($\bar{v}(x)$) and associated frequency (ω) as

$$\bar{v}''''(x) - \frac{\rho A \omega^2}{EI}\,\bar{v}(x) = 0. \tag{8.9}$$

Introduce the parameter β (dimensions: length^{-1}) by

$$\beta^4 \equiv \frac{\rho\omega^2 A}{EI}; \tag{8.10}$$

the differential equation (8.9) is then

$$\bar{v}''''(x) - \beta^4 \bar{v}(x) = 0. \tag{8.11}$$

(3.) The linear, ordinary differential equation with constant coefficients (8.11) has homogeneous solutions [2] of the form

$$\bar{v}(x) = C_1 \cosh\beta x + C_2 \sinh\beta x + C_3 \cos\beta x + C_4 \sin\beta x, \tag{8.12}$$

for to-be-determined constants C_1, C_2, C_3, and C_4. In writing (8.12), we make use of the well-known equivalence of linear combinations of exponentials and hyperbolic functions ($2\cosh z = e^z + e^{-z}$, etc.) and Euler's equation ($2\cos x = e^{ix} + e^{-ix}$, etc.) to choose trigonometric and hyperbolic trig functions as the base functions for the solution (rather than real and imaginary exponential functions $e^{\beta x}$, etc.).

(4.) The kinematic (geometric) boundary conditions of zero slope and displacement (for all time) at $x = 0$ require $\bar{v}(x = 0) = 0$ and $\bar{v}'(x = 0) = 0$; when these conditions are imposed on solutions of the form (8.12), one obtains, respectively,

$$C_1 + C_3 = 0 \Rightarrow C_3 = -C_1; \tag{8.13a}$$

$$C_2 + C_4 = 0 \Rightarrow C_4 = -C_2. \tag{8.13b}$$

Thus the general mode shape can be expressed in terms of two integration constants, C_1 and C_2, as

$$\bar{v}(x) = C_1\,(\cosh\beta x - \cos\beta x) + C_2(\sinh\beta x - \sin\beta x). \tag{8.14}$$

The dynamic ('force') boundary conditions at $x = L$ are, for all times, $M(x = L, t) = 0$ and $V(x = L, t) = 0$, respectively. In terms of the solution, these in turn require $\bar{v}''(x = L) = 0$ and $\bar{v}'''(x = L) = 0$, respectively. Using matrix notation to write out these last two equations as linear combinations of the coefficients C_1 and C_2, there results (after factoring out the constant factor EI and all common powers of β):

$$\begin{bmatrix} (\cosh\beta L + \cos\beta L) & (\sinh\beta L + \sin\beta L) \\ (\sinh\beta L - \sin\beta L) & (\cosh\beta L + \cos\beta L) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{8.15}$$

[2] Try $\bar{v}(x) = \exp\lambda x$; then the trial form satisfies the differential equation if $\lambda^4 = \beta^4$, so that the characteristic roots satisfy $\lambda^2 = \pm\beta^2$, leading to the four characteristic roots as $\lambda = \pm\beta\quad$ or $\pm\, i\beta$, etc.

In order for the matrix equation (8.15) to have a non-trivial solution, it is necessary for the determinant of the 2 by 2 matrix to vanish, requiring

$$(\cosh\beta L + \cos\beta L)^2 = (\sinh\beta L + \sin\beta L)(\sinh\beta L - \sin\beta L) = \sinh^2\beta L - \sin^2\beta L. \tag{8.16}$$

Rearranging the algebra in (8.16), and using trig and hyperbolic trig identities, gives

$$(\cos^2\beta L + \sin^2\beta L) + 2\cos\beta L\cosh\beta L = \sinh^2\beta L - \cosh^2\beta L \equiv -1, \tag{8.17}$$

or

$$2\ (1 + \cos\beta L\cosh\beta L) = 0. \tag{8.18}$$

Now define the dimensionless parameter $\xi \equiv \beta L$; roots of (8.18) occur only when $\cos\xi\cosh\xi = -1$. There are an infinite number of ξ-values satisfying this relation; the smallest (corresponding to the lowest natural frequency) is at $\xi = \beta L = 1.875104 \equiv \xi^\star$. Recalling the definition of β, we note that

$$(\beta L)^4 = (\xi)^4 = \frac{\omega^2\rho A L^4}{EI}. \tag{8.19}$$

(5.) Thus, the lowest natural frequency, $\omega = \omega_1$, corresponding to the smallest root, $\xi = \xi_1 = \beta_1 L \equiv \xi^\star$, of (8.18) is

$$\omega_1 = \left(\frac{\xi_1}{L}\right)^2\sqrt{\frac{EI}{\rho A}}. \tag{8.20}$$

As to the mode shape of the first mode, note that since the determinant in (8.15) equals zero, for the chosen natural frequency, the ratio of the two coefficients C_1 and C_2 is fixed at, for example,

$$C_2 = -\left[\frac{(\cosh\beta L + \cos\beta L)}{(\sinh\beta L + \sin\beta L)}\right] C_1 \equiv R\,C_1, \tag{8.21}$$

where the dimensionless ratio "R" has been introduced. Returning to (8.14), and using (8.21), the fundamental mode shape can finally be given by

$$\bar{v}(x) = C_1\ [(\cosh\beta x - \cos\beta x) + R\,(\sinh\beta x - \sin\beta x)]\,, \tag{8.22}$$

for some constant C_1. The final scaling factor for the vibration, C_1, is undetermined from the analysis, but it is understood that it remains sufficiently small so that only linear elastic deformation occurs, and small enough that the maximum lateral displacements and rotations of the beam remain "small".

While we focus in the lab on the first mode of vibration, there are, as noted, an infinite number of vibration frequencies, ω_i, and corresponding vibration modes, $\bar{v}_i(x)$. Each frequency is given, in turn, by

$$\omega_i = \left(\frac{\xi_i}{L}\right)^2\sqrt{\frac{EI}{\rho A}}, \tag{8.23}$$

where ξ_i is the 'ith' root of (8.18), and the ith mode shape is

$$\bar{v}_i(x) = C_{1(i)} \left[(\cosh \beta_i x - \cos \beta_i x) + R_i (\sinh \beta_i x - \sin \beta_i x) \right], \qquad (8.24),$$

where $\beta_i \equiv \xi_i / L$, $C_{1(i)}$ is an arbitrary scale factor for the amplitude of the ith vibration mode, and

$$R_i \equiv \frac{(\cosh \xi_i + \cos \xi_i)}{(\sinh \xi_i + \sin \xi_i)}. \qquad (8.25)$$