# Introduction to Numerical Analysis for Engineers

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# Numerical Interpolation

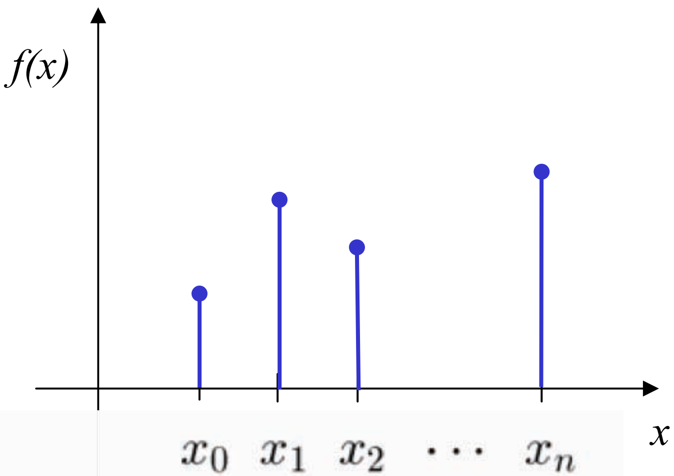
Given: $f(x_0) = f_0\ ,\ f(x_1) = f_1\ , \ldots f(x_n) = f_n$

Find $f(x)$ for $x \in [x_0, x_n]$

## Purpose of numerical Interpolation

1. Compute intermediate values of a sampled function
2. Numerical differentiation – foundation for Finite Difference and Finite Element methods
3. Numerical Integration

# Numerical Interpolation
# Polynomial Interpolation




Interpolation

$$
\begin{aligned}
f(x) &\simeq F(x) \\
f(x_i) &= F(x_i)
\end{aligned}
$$

$F(x)$ Interpolation function

## Polynomial Interpolation

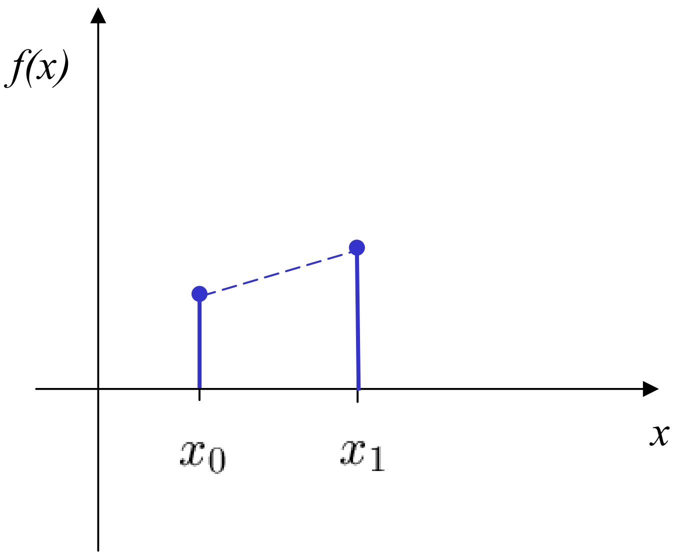
$$F(x) = p(x) = a_0 x^n + a_1 x^{n-1} \cdots a_{n-1} x + a_n$$

## Coefficients: Linear System of Equations

$$
\begin{aligned}
f_0 &= a_0 x_0^n + a_1 x_0^{n-1} \cdots a_{n-1} x_0 + a_n \\
f_1 &= a_0 x_1^n + a_1 x_1^{n-1} \cdots a_{n-1} x_1 + a_n \\
&\cdot \\
&\cdot \\
&\cdot \\
f_n &= a_0 x_n^n + a_1 x_n^{n-1} \cdots a_{n-1} x_n + a_n
\end{aligned}
$$

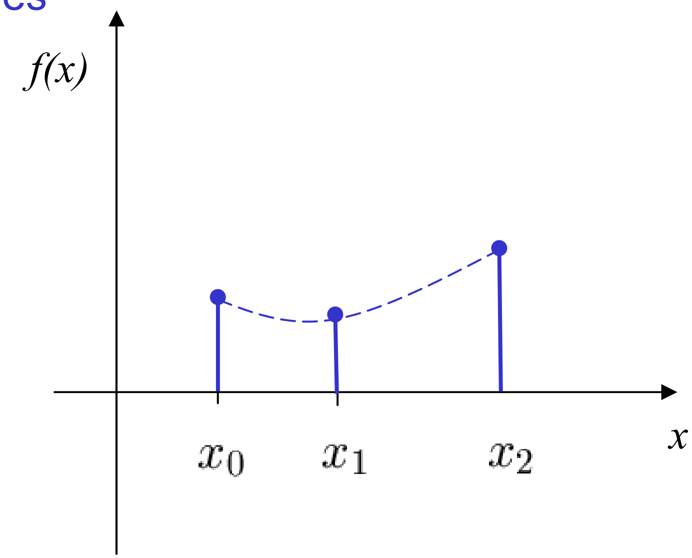
# Numerical Interpolation
# Polynomial Interpolation

## Examples




Linear Interpolation

$$p(x) = f_0 + (f_1 - f_0)\frac{x - x_0}{x_1 - x_0}$$




Quadratic Interpolation

$$p(x) = a_0 x^2 + a_1 x + a_2$$

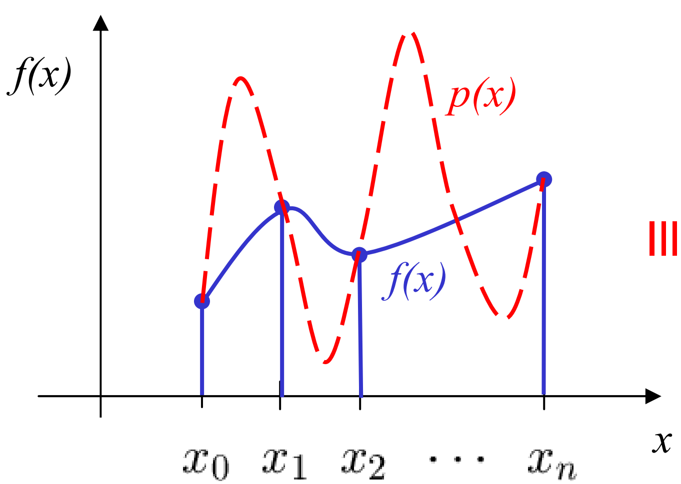
# Numerical Interpolation
# Polynomial Interpolation

## Taylor Series

$$f(x) = p(x) + r(x) = f(x_0) + \sum_{i=1}^{n} \frac{f^{(i)}(x_0)}{(i+1)!}(x-x_0)^i + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

Remainder

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$




Requirement

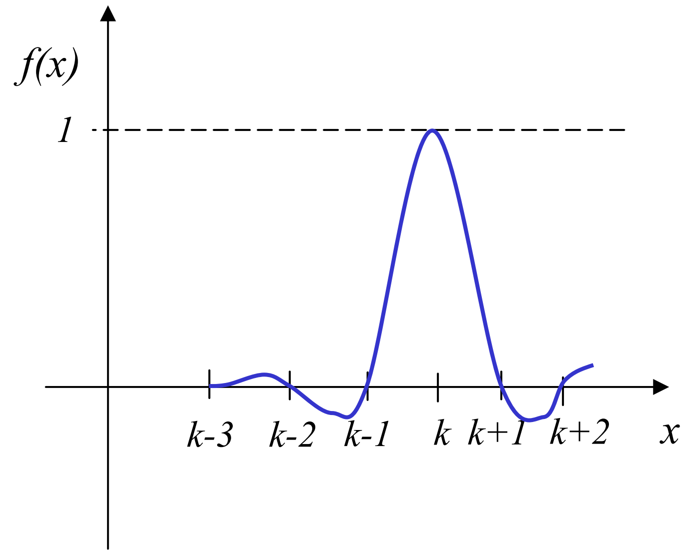
$$f^{(n+1)}(\xi) \ll 1$$

Ill-conditioned for large n

Polynomial is unique, but how do we calculate the coefficients?

# Numerical Interpolation
# Lagrange Polynomials

$$p(x) = \sum_{k=0}^{n} L_k(x) f(x_k) = \sum_{k=0}^{n} L_k(x) f_k$$




$$L_k(x) = \sum_{i=0}^{n} \ell_{ik} x^i$$

$$L_k(x_i) = \delta_{ki} = \begin{cases} 0 & k \neq i \\ 1 & k = i \end{cases}$$

$$L_k(x) = \prod_{j=0, j\neq k}^{n} \frac{x - x_j}{x_k - x_j}$$

Difficult to program
Difficult to estimate errors
Divisions are expensive

Important for numerical integration

# Numerical Interpolation
# Triangular Families of Polynomials

## Ordered Polynimials

$$p(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x)$$

where

$$\begin{aligned}
\phi_0(x) &= a_{00} \\
\phi_1(x) &= a_{10} + a_{11}x \\
\phi_1(x) &= a_{20} + a_{21}x + a_{22}x^2 \\
&\cdot \\
&\cdot \\
\phi_n(x) &= a_{n0} + a_{n1}x + \cdots\cdots\cdots + a_{nn}x^n
\end{aligned}$$

## Special form convenient for interpolation

$$\begin{aligned}
\phi_0(x) &= 1 \\
\phi_1(x) &= x - x_0 \\
\phi_2(x) &= (x - x_0)(x - x_1) \\
&\cdot \\
&\cdot \\
\phi_n(x) &= (x - x_0)(x - x_1)\cdots(x - x_{n-1})
\end{aligned}$$

## Coefficients

$$\begin{aligned}
f(x_0) = p(x_0) &= c_0 \\
f(x_1) = p(x_1) &= c_0 + c_1(x_1 - x_0) \\
f(x_2) = p(x_2) &= c_0 + c_1(x_2 - x_0) + c_2(x_2 - x_0)
\end{aligned}$$

$c_0, \; c_1, \; \ldots c_n$ found by recursion

$\cdot$

# Numerical Interpolation
# Triangular Families of Polynomials

## Polynomial Evaluation
## Horner's Scheme

$$
\begin{aligned}
f(x) &\simeq c_0\phi_0(x) + c_1\phi_1(x)\cdots \\
&= c_0 + (x - x_0)(c_1 + (x - x_1)(c_2 + (x - x_2)(\cdots)))
\end{aligned}
$$

Remainder – Interpolation Error

$$
r(x) = f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)\cdots(x - x_n)
$$

# Numerical Interpolation
# Newton's Iteration Formula

## Standard triangular family of polynomials

$$
\begin{aligned}
f(x) &= p(x) + r(x) \\
&= c_0 + c_1(x - x_0)\cdots + c_n(x - x_0)\cdots(x - x_{n-1}) \\
&\quad + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)\cdots(x - x_n)
\end{aligned}
$$

## Divided Differences

$$f(x_0) = c_0 \Rightarrow c_0 = \boxed{f(x_0)}$$

$$f(x_1) = c_0 + c_1(x_1 - x_0) \Rightarrow c_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \boxed{f[x_0, x_1]}$$

$$f(x_2) = c_0 + c_1(x_2 - x_0) + c_2(x_2 - x_0)(x_2 - x_1)$$

$$c_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0} = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \boxed{f[x_0, x_1, x_2]}$$

$$c_n = \boxed{f[x_0, x_1, \ldots, x_n]} = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots x_{n-1}]}{x_n - x_0}$$

## Newton's Computational Scheme

$c_0$: $x_0 \quad f(x_0)$

$c_1$: $f[x_0, x_1]$

$x_1 \quad f(x_1)$

$c_2$: $f[x_0, x_1, x_2]$

$f[x_1, x_2]$

$c_3$: $f[x_0, x_1, x_2, x_3]$

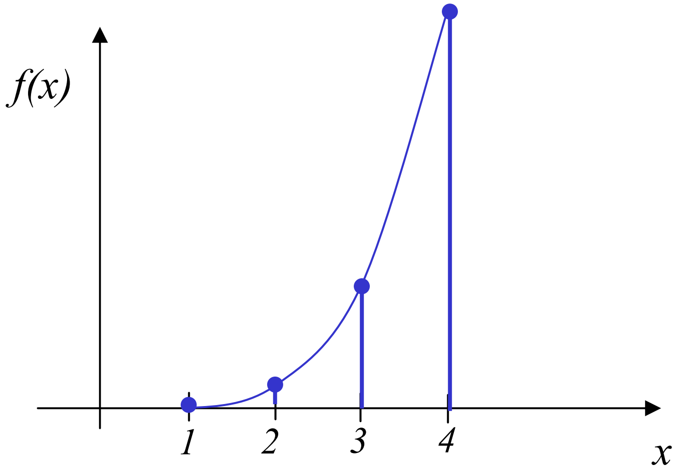
$x_2 \quad f(x_2)$

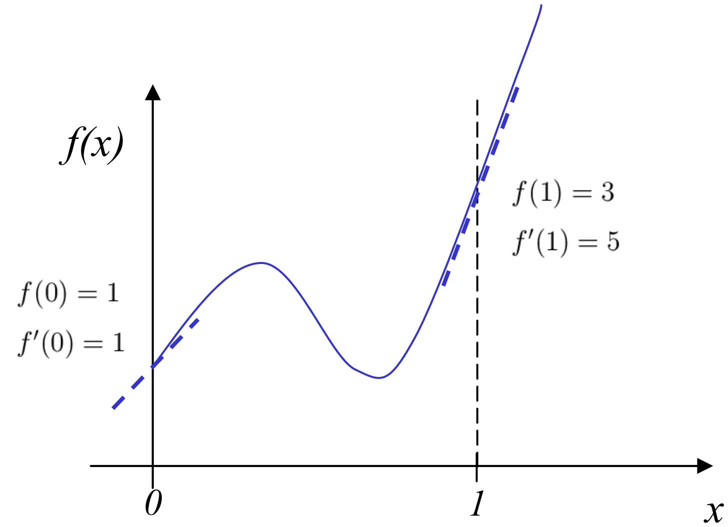
$f[x_1, x_2, x_3]$

$f[x_2, x_3]$

$x_3 \quad f(x_3)$

# Numerical Interpolation
# Newton's Iteration Formula




| $i$ | $x_i$ | $f(x_i)$ | | | |
|---|---|---|---|---|---|
| 0 | 1 | 0 | | | |
| | | | $(4-0)/1 = 4$ | | |
| 1 | 2 | 4 | | $(16-4)/2 = 6$ | |
| | | | $(20-4)/1 = 16$ | | $(12-6)/3 = 2$ |
| 2 | 3 | 20 | | $(40-16)/2 = 12$ | |
| | | | $(60-20)/1 = 40$ | | |
| 3 | 4 | 60 | | | |

$$p(x) = 4(x-1) + 6(x-1)(x-2) + 2(x-1)(x-2)(x-3)$$

$$\lim_{\epsilon \to 0} f[x_0, x_0 + \epsilon] = f'(x_0)$$

| $i$ | $x_i$ | $f(x_i)$ | | | |
|---|---|---|---|---|---|
| 0 | 0 | 1 | | | |
| | | | $f'(0) = 1$ | | |
| 1 | $0(+\epsilon)$ | $1(+\epsilon f'(0))$ | | $(2-1)/1 = 1$ | |
| | | | $(3-1)/1 = 2$ | | $(3-1)/1 = 2$ |
| 2 | $1(-\epsilon)$ | $3(-\epsilon f'(1))$ | | $(5-2)/1 = 3$ | |
| | | | $f'(1) = 5$ | | |
| 3 | 1 | 60 | | | |

$$p(x) = 1 + x + x^2 + 2x^2(x-1) = 1 + x - x^2 + 2x^3$$

# Numerical Interpolation
# Equidistant Newton Interpolation

## Equidistant Sampling

$$x_i = x_0 + ih$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1}{h}(f_1 - f_0) = \frac{1}{h}\Delta f_0$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$= \frac{1}{1 \cdot 2 \cdot h^2}(f_2 - 2f_1 + f_0) = \frac{1}{2!h^2}\Delta^2 f_0$$

$$f[x_0, x_1, x_2, x_3] = \frac{1}{3! \cdot h^3}(f_3 - 3f_2 + 3f_1 - f_0) = \frac{1}{3!h^3}\Delta^3 f_0$$

## Divided Differences
## Stepsize Implied

$x_0 \quad f_0$

$f_1 - f_0 = \Delta f_0$

$x_1 \quad f_1$

$f_2 - 2f_1 + f_0 = \Delta^2 f_0$

$f_2 - f_1 = \Delta f_1$

$f_3 - 3f_2 + 3f_1 - f_0 - = \Delta^3 f_0$

$x_2 \quad f_2$

$f_3 - 2f_2 + f_1 = \Delta^2 f_1$

$f_3 - f_2 = \Delta f_2$

$x_3 \quad f_3$

$$c_0 = f_0$$

$$c_1 = \frac{1}{h}\Delta f_0$$

$$c_2 = \frac{1}{2!h^2}\Delta^2 f_0$$

$$c_3 = \frac{1}{3!h^3}\Delta^3 f_0$$

# Numerical Interpolation
# Newton's Iteration Formula

```
function[a] = interp_test(n)
%n=2
h=1/n
xi=[0:h:1]
f=sqrt(1-xi.*xi) .* (1 - 2*xi +5*(xi.*xi));
%f=1-2*xi+5*(xi.*xi)-4*(xi.*xi.*xi);

c=newton_coef(h,f)
m=101
x=[0:1/(m-1):1];
fx=sqrt(1-x.*x) .* (1 - 2*x +5*(x.*x));
%fx=1-2*x+5*(x.*x)-4*(x.*x.*x);

y=newton(x,xi,c);
hold off; b=plot(x,fx,'b'); set(b,'LineWidth',2);
hold on; b=plot(xi,f,'.r') ; set(b,'MarkerSize',30);
b=plot(x,y,'g'); set(b,'LineWidth',2);
yl=lagrange(x,xi,f);
b=plot(x,yl,'xm'); set(b,'Markersize',5);
b=legend('Exact','Samples','Newton','Lagrange')
b=title(['n = ' num2str(n)]); set(b,'FontSize',16);
```

```
function[y] = newton(x,xi,c)
% Computes Newton polynomial
% with coefficients c
n=length(c)-1
m=length(x)
y=c(n+1)*ones(1,m);
for i=n-1:-1:0
    cc=c(i+1);
    xx=xi(i+1);
    y=cc+y.*(x-xx);
end
```

```
function[c] = newton_coef(h,f)
% Computes Newton Coefficients
% for equidistant sampling h
n=length(f)-1
c=f; c_old=f; fac=1;
for i=1:n
fac=i*h;
for j=i:n
 c(j+1)=(c_old(j+1)-c_old(j))/fac;
end
c_old=c;
end
```