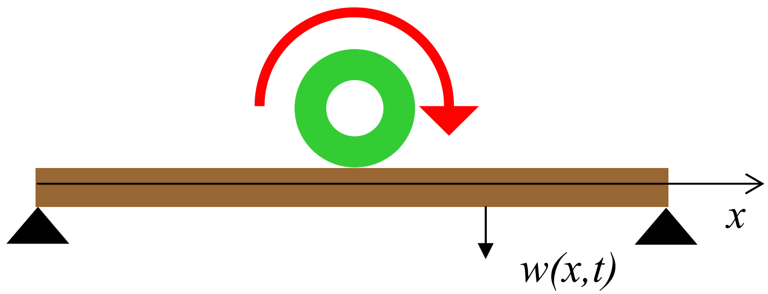
Introduction to Numerical Analysis for Engineers

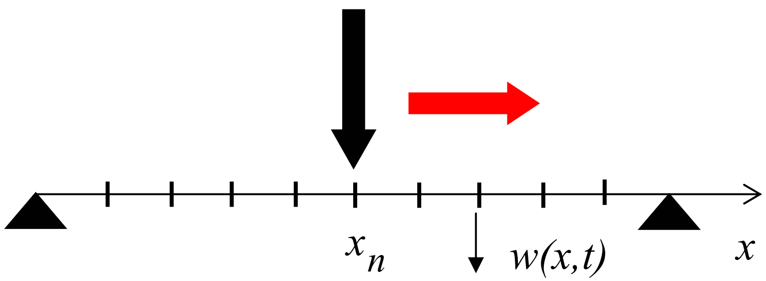
- Fundamentals of Digital Computing
  - Digital Computer Models
  - Convergence, accuracy and stability
  - Number representation
  - Arithmetic operations
  - Recursion algorithms
- Error Analysis
  - Error propagation – numerical stability
  - Error estimation
  - Error cancellation
  - Condition numbers

Digital Computer Models

Continuous Model



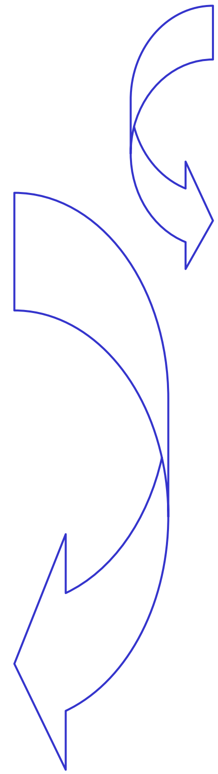
Discrete Model



$$t_m = t_0 + m\Delta t \ , \quad m = 0, 1, \ldots M-1$$

$$x_n = x_0 + n\,\Delta x \ , \quad n = 0, 1, \ldots N-1$$

$$\frac{dw}{dx} \simeq \frac{\Delta w}{\Delta x} \ , \quad \frac{dw}{dt} \simeq \frac{\Delta w}{\Delta t}$$



Differential Equation

$$L(p, w, x, t) = 0$$

Differentiation
Integration

Difference Equation

$$L_{mn}(p_{mn}, w_{mn}, x_n, t_m) = 0$$

System of Equations

$$\sum_{j=0}^{N-1} F_i(w_j) = B_i$$

Linear System of Equations

$$\sum_{j=0}^{N-1} A_{ij} w_j = B_i$$

Solving linear equations

Eigenvalue Problems

$$\overline{\overline{\mathbf{A}}}\mathbf{u} = \lambda \mathbf{u} \Leftrightarrow (\overline{\overline{\mathbf{A}}} - \lambda \overline{\overline{\mathbf{I}}})\mathbf{u} = \mathbf{0}$$

Non-trivial Solutions

$$\det(\overline{\overline{\mathbf{A}}} - \lambda \overline{\overline{\mathbf{I}}}) = 0$$

Root finding

Accuracy and Stability => Convergence

Floating Number Representation

$$r = mb^e$$

m	Mantissa
b	Base
e	Exponent

Examples

Decimal $\quad 0.00527 = 0.527_{10} \times 10^{-2_{10}}$

Binary $\quad 10.1_2 = 0.101_2 \times 2^{2_{10}} = 0.101_2 \times 2^{10_2}$

Convention

Decimal $\quad 0.1 \leq m < 1.0$

Binary $\quad 0.1_2 = 0.5_{10} \leq m < 1.0$

General $\quad b^{-1} \leq m < b^0$

Max mantissa $\quad 0.11\ldots1 = 0.999999$

Min mantissa $\quad 0.10\ldots0 = 0.5$

Max exponent $\quad 2^7 - 1 = 127 \quad 2^{127} \simeq 1,7 \times 10^{38}$

Min exponent $\quad -2^7 = -128 \quad 2^{-128} \simeq 2.9 \times 10^{-39}$

Arithmetic Operations

Number Representation

Absolute Error

$$\bar{\epsilon} = |\bar{m} - m| \leq \frac{1}{2} b^{-t}$$

Relative Error

$$\bar{\alpha} = \frac{|\bar{m} - m| b^e}{|m| b^e} \leq \frac{\frac{1}{2} b^{-t}}{b^{-1}} \leq \frac{1}{2} b^{1-t}$$

Addition and Subtraction

$$r_1 \pm r_2 = m_1 b^{e_1} \pm m_2 b^{e_2}$$

Shift mantissa of largest number

$$e_1 > e_2$$

Result has exponent of largest number

$$r_1 \pm r_2 = \left(m_1 \pm m_2 b^{e_2 - e_1}\right) b^{e_1} = m b^{e1}$$

Absolute Error

$$\bar{\epsilon} \leq \bar{\epsilon_1} + \bar{\epsilon_2}$$

Relative Error

$$\bar{\alpha} = \frac{|\bar{m} - m|}{|m|}$$

Unbounded

Multiplication and Division

$$r_1 \times r_2 = m_1 m_2 b^{e1+e2}$$

$$m = m1m2 < 1$$

$$0.1_2 \times 0.1_2 = 0.01_2$$

Relative Error

$$\bar{\alpha} \leq \bar{\alpha_1} + \bar{\alpha_2}$$

Bounded

Recursion

Numerically evaluate square-root

$$\sqrt{s}\,, \quad s > 0$$

Initial guess

$$x_0 \simeq \sqrt{x}$$

Test

$$x_0^2 < s \quad \Rightarrow \quad x_0 < \sqrt{x} \quad \Rightarrow \quad \frac{1}{x_0} > \frac{1}{\sqrt{x}}$$

$$x_0^2 < s \quad \Rightarrow \quad x_0 < \sqrt{x} \quad \Rightarrow \quad \frac{1}{x_0} > \frac{1}{\sqrt{x}}$$

Mean of guess and its reciprocal

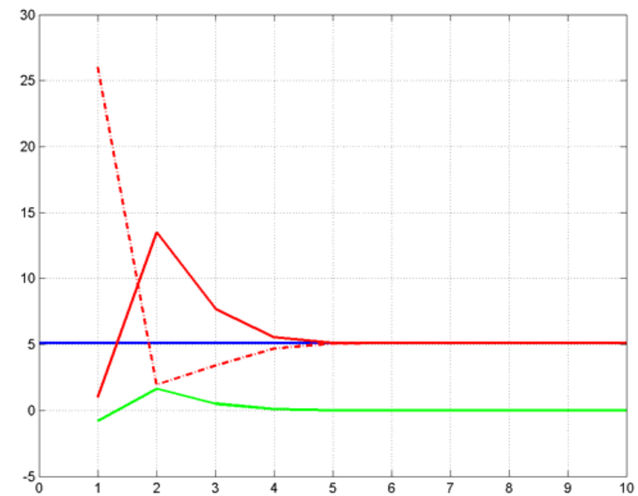
$$x_1 = \frac{1}{2}\left(x_0 + \frac{1}{x_0}\right)$$

Recursion Algorithm

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right)$$

MATLAB script
sqr.m

```
a=26;
n=10;
g=1;
     sq(1)=g;
     for i=2:n
      sq(i)= 0.5*(sq(i-1) + a/sq(i-1));
     end
     hold off
     plot([0 n],[sqrt(a) sqrt(a)],'b')
     hold on
     plot(sq,'r')
     plot(a./sq,'r-.')
     plot((sq-sqrt(a))/sqrt(a),'g')
     grid on
```



Recursion
Horner's Scheme

Evaluate polynomial

$$
\begin{aligned}
p(z) &= a_0 z^3 + a_1 z^2 + a_2 z + a_3 \\
&= \left((a_0 z + a_1)z + a_2\right) z + a_3
\end{aligned}
$$

Horner's Scheme

a_0	a_1	a_2	a_3
	zb_0	zb_1	zb_2
b_0	b_1	b_2	b_3

$$p(z) = b_3$$

General order n

$$p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

Recurrence relation

$$b_0 = a_0 \,, \quad b_i = a_i + z b_{i-1} \,, \quad i = 1, \ldots n$$

$$\boxed{p(z) = b_n}$$

horner.m

```
% Horners scheme
% for evaluating polynomials
a=[ 1 2 3 4 5 6 7 8 9 10 ];
n=length(a) -1 ;
z=1;
b=a(1);
% Note index shift for a
for i=1:n
    b=a(i+1)+ z*b;
end
p=b
```

```
>> horner

p =

    55

>>
```

Recursion

Order of Operations Matter

0 1

$$y = f(x) = \sum_{n=1}^{\infty} [x^n + b\sin[\pi/2 - \pi/10n] - c\cos[pi/(10(n+1))]]$$

$$x = 0.5\ , \quad b = 0\ , \quad c = 0 \Rightarrow y = 1.0$$

Result of small, but significant term ‘destroyed’ by subsequent addition and subtraction of almost equal, large numbers.

Remedy:
Change order of additions

recur.m

```
N=20; sum=0; sumr=0;
b=1; c=1; x=0.5;
xn=1;
% Number of significant digits in computations
dig=2;
ndiv=10;
for i=1:N
a1=sin(pi/2-pi/(ndiv*i));
a2=-cos(pi/(ndiv*(i+1)));
% Full matlab precision
xn=xn*x;
addr=xn+b*a1;
addr=addr+c*a2;
ar(i)=addr;
sumr=sumr+addr;
z(i)=sumr;
% additions with dig significant digits
add=radd(xn,b*a1,dig);
add=radd(add,c*a2,dig);
% add=radd(b*a1,c*a2,dig);
% add=radd(add,xn,dig);
a(i)=add;
sum=radd(sum,add,dig);
y(i)=sum;
end
sumr
'     i        delta       Sum     delta(approx) Sum(approx)'
res=[[1:1:N]' ar' z' a' y']

hold off
a=plot(y,'b'); set(a,'LineWidth',2);
hold on
a=plot(z,'r'); set(a,'LineWidth',2);
a=plot(abs(z-y)./z,'g'); set(a,'LineWidth',2);
legend([ num2str(dig) ' digits'],'Exact','Error');
```

recur.m

```
>> recur

b = 1; c = 1; x = 0.5;
dig=2

      i        delta       Sum     delta(approx) Sum(approx)


res =

    1.0000    0.4634    0.4634    0.5000    0.5000
    2.0000    0.2432    0.7065    0.2000    0.7000
    3.0000    0.1226    0.8291    0.1000    0.8000
    4.0000    0.0614    0.8905    0.1000    0.9000
    5.0000    0.0306    0.9212         0    0.9000
    6.0000    0.0153    0.9364         0    0.9000
    7.0000    0.0076    0.9440         0    0.9000
    8.0000    0.0037    0.9478         0    0.9000
    9.0000    0.0018    0.9496         0    0.9000
   10.0000    0.0009    0.9505         0    0.9000
   11.0000    0.0004    0.9509         0    0.9000
   12.0000    0.0002    0.9511         0    0.9000
   13.0000    0.0001    0.9512         0    0.9000
   14.0000    0.0000    0.9512         0    0.9000
   15.0000    0.0000    0.9512         0    0.9000
   16.0000   -0.0000    0.9512         0    0.9000
   17.0000   -0.0000    0.9512         0    0.9000
   18.0000   -0.0000    0.9512         0    0.9000
   19.0000   -0.0000    0.9512         0    0.9000
   20.0000   -0.0000    0.9512         0    0.9000
```