

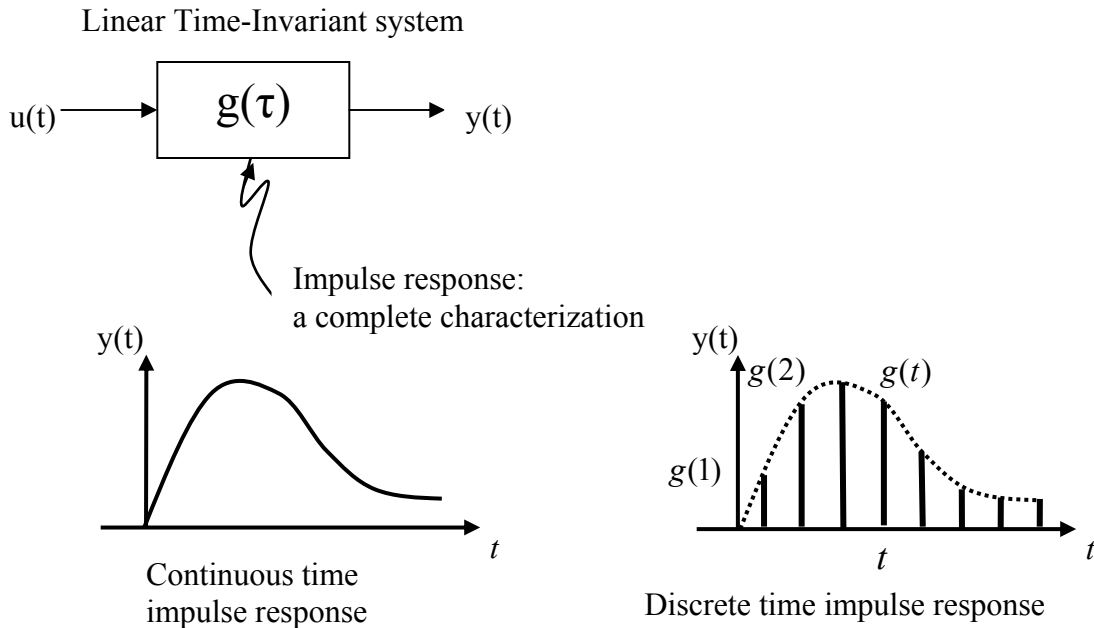
**2.160 System Identification, Estimation, and Learning**  
**Lecture Notes No. 9**  
**March 8, 2006**

***Part 2 Representation and Learning***

We now move on to the second part of the course, Representation and Learning. You will learn various forms of system representation, including linear and nonlinear systems. We will use “Prediction” form as a generic representation of diverse systems. You will also learn data compression and learning, which are closely related to system representation and prediction.

**5 Prediction Modeling of Linear Systems**

**5.1 Impulse Response and Transfer Operator (Review)**



Any linear time-invariant system can be characterized completely with Impulse Response:  $g(t)$ . For an arbitrary input,  $\{u(s)|s \leq t\}$ , the output  $y(t)$  is given by the convolution of the input and the impulse response given by

Continuous Time Convolution 
$$y(t) = \int_0^{\infty} g(\tau)u(t - \tau)d\tau \tag{1}$$

For discrete-time systems, an impulse response is given by an infinite time series:

$g(0), g(1), g(2), \dots$ , or  $\{g(\tau)|\tau = 0 \dots \infty\}$

$\{u(s)|s \leq t\} \Rightarrow$  Output  $\{y(s)|s \leq t\}$

Discrete Time Convolution

$$y(t) = \sum_{k=1}^{\infty} g(k)u(t-k) \quad (2)$$

Forward shift operator:  $q$

$$qu(t) = u(t+1) \quad (3)$$

Backward shift operator:  $q^{-1}$

$$q^{-1}u(t) = u(t-1) \quad (4)$$

Continuous Time Differential Operator

$$p = \frac{d}{dt}$$

$$\ddot{y} + 2\dot{y} + 3y = \dot{u} + 2u$$

$$(p^2 + 2p + 3)y(t) = (p + 2)u(t)$$

$$y(t) = \frac{p+2}{p^2 + 2p + 3}u(t)$$

Using(3) (4) in (2)

$$y(t) = \sum_{k=1}^{\infty} g(k)(q^{-k}u(t)) \quad (5)$$

$$= \sum_{k=1}^{\infty} (g(k)q^{-k})u(t) = G(q)u(t)$$

$\parallel$  Transfer Operator or  
G(q) Transfer Function

Monic Transfer Function:  $G(q) = 1 + \sum_{k=1}^{\infty} g(k)q^{-k} = \sum_{k=0}^{\infty} g(k)q^{-k}$  w/  $g(0) = 1$

## 5.2 Z-Transform:

Taking Laplace transform of (2) yields

$$L[y] = \sum g(k) \underbrace{e^{-k\Delta T s}}_{(e^{\Delta T s})^{-k}} L[u]$$

where  $\Delta T$  is a sampling interval. Replacing  $e^{\Delta T s}$  by  $z$  yields the z-transform of the transfer function

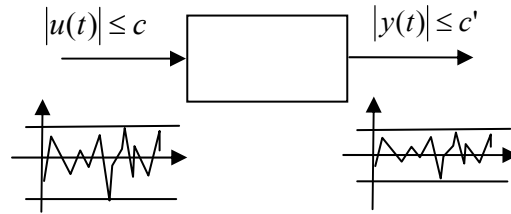
$$G(z) = \sum_{k=1}^{\infty} g(k)z^{-k} \quad (6)$$

which is a complex function of  $z = e^{\Delta T s}$ . Poles and zeros are defined as:

zero: A zero of  $G(z)$  is a complex number  $z_i$  that makes  $G(z)$  zero:  $G(z_i) = 0$

pole: A pole of  $G(z)$  is a complex number  $z_j$  that makes  $G(z)$  infinite  
 $G(z_j) = \infty$

## Bounded-input, bounded output (BIBO) stability

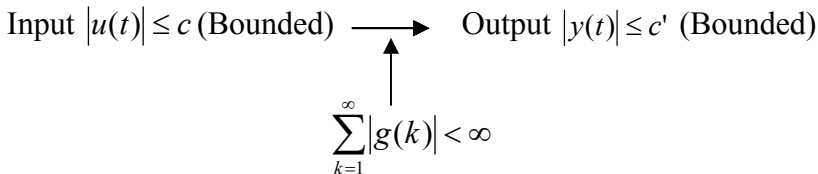


The transfer function  $G(q) = \sum_{k=1}^{\infty} g(k)q^{-k}$  is said to be stable if

$$\sum_{k=1}^{\infty} |g(k)| < \infty \quad (7)$$

$$|y(t)| = \left| \sum_{k=1}^{\infty} g(k)u(t-k) \right| \leq \sum_{k=1}^{\infty} |g(k)u(t-k)| \leq c \sum_{k=1}^{\infty} |g(k)| \leq c'$$

Therefore, the condition of (7) satisfies the BIBO stability criterion.



In the z-domain, this stability is interpreted as follows. Consider (6) as a Laurent expansion

$$G(z) = \sum_{k=1}^{\infty} g(k)z^{-k}$$

Under the condition of (7), this is convergent for all  $|z| \geq 1$

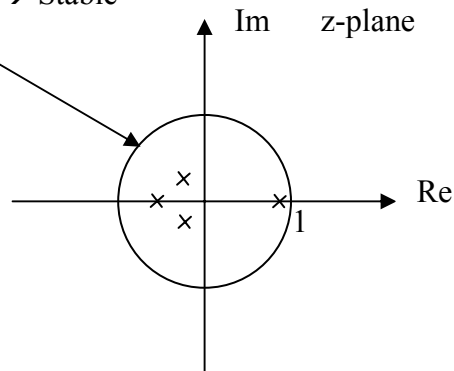


Function  $G(z)$  is analytic on and outside the unit circle

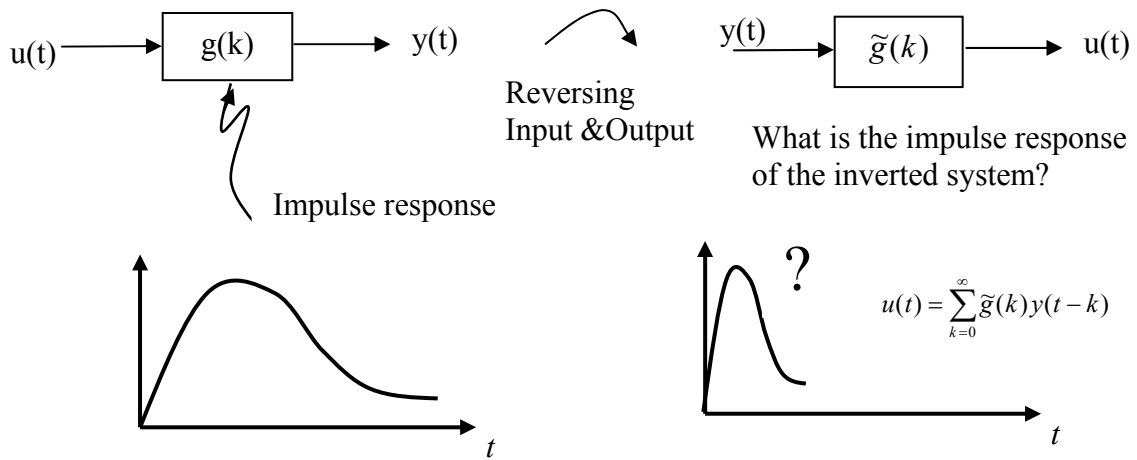


$G(z)$  has no poles on and outside the unit circle. → Stable

If poles exist, they must be within the unit circle.



## Inverse Transfer Function



Consider the  $z$ -transform  $G(z) = \sum_{k=0}^{\infty} g(k)z^{-k}$  is stable,

Assume that function  $\frac{1}{G(z)}$  is analytic in  $|z| \geq 1$ ,

Then it can be expressed in Laurent expansion form

$$\frac{1}{G(z)} = \sum_{k=0}^{\infty} \tilde{g}(k)z^{-k} \quad (8)$$

This  $\tilde{g}(k) \quad k=0 \dots \infty$  gives the impulse response of the inverse system. We write

$$G^{-1}(q) = \sum_{k=0}^{\infty} \tilde{g}(k)q^{-k} \quad (9)$$

Example:

$$\begin{aligned} y(t) = u(t) + cu(t-1) &\longrightarrow y(t) = u(t) + cq^{-1}u(t) \\ G(q) = 1 + cq^{-1} &= (1 + cq^{-1})u(t) \end{aligned}$$

$z$ -transform  $G(z) = 1 + cz^{-1} = \frac{z+c}{z}$

Inverting  $G(z)$

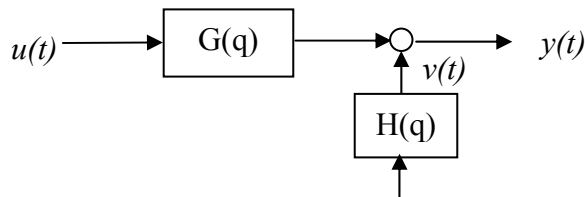
$$G^{-1}(z) = \frac{1}{G(z)} = \frac{z}{z+c} = \frac{1}{1+cz^{-1}} = \sum_{k=0}^{\infty} (-c)^k z^{-k}$$

Therefore  $u(t) = \sum_{k=0}^{\infty} (-c)^k y(t-k)$

Note that we can compute  $\frac{1}{G(z)}$ , since  $G(z)$  is a function of complex number, not a function of “operator”.

### 5.3 Noise Dynamics

In Kalman filter, two separate noise sources, i.e. process noise and measurement noise, were considered. However, identifying the two noise characteristics separately is difficult or infeasible in some class of practical applications. In the following we will consider an aggregated noise model originated in a single random process.



To model  $v(t)$

We assume that  $v(t)$  comes from a dynamic process  $H(q)$  driven by a totally unpredictable random process  $e(t)$

$H(q)$  = deterministic; to be identified

$e(t)$  = uncorrelated, zero mean

$$E[e(t)] = 0 \quad E[e(t)e(s)] = \begin{cases} 0 & t \neq s \\ \lambda & t = s \end{cases} \quad (10)$$

$$y(t) = G(q)u(t) + H(q)e(t) \quad (11)$$

System Identification: Determine  $G(q)$  and  $H(q)$  from input-output data

### 5.4 Prediction

$$y(t) = G(q)u(t) + H(q)e(t)$$

If we know  $G(q)$  and  $H(q)$ , how can we best predict output  $y(t)$  based on output observations up to  $t-1$ ?

This prediction problem is equivalent to predict  $v(t)$

$$v(t) \equiv H(q)e(t) = y(t) - G(q)u(t) \quad (12)$$

based on observation of  $v$  up to  $t-1$ ,  $\{v(s) | s \leq t-1\}$