Session 9

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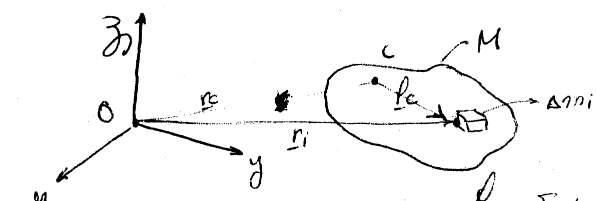
$\underline{H}_c = \underline{\underline{I}}_c \underline{\omega}$
↳ moment of inertia tensor

$\underline{H}_B = \underline{\underline{I}}_B \underline{\omega}$

(3) Work-Energy Principle



$$T = \sum_{i=1}^{N} \frac{1}{2} \Delta m_i |\dot{r}_i|^2 \underset{N\to\infty}{=} \frac{1}{2}\int_M |\underline{v}|^2 dm$$

also using the limit of the argument given for systems of particles, $W_{12} = T_2 - T_1$

work done by external forces (rigid body)

If all forces are potential,

$T+V = \text{Const}$ Conservation of energy

Conservation of Energy

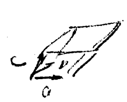
To evaluate T let $\underline{v} = \underline{v}_c + \underline{\omega} \times \underline{\rho}_c$

$$\Rightarrow T = \frac{1}{2}\int_M [\underline{v}_c + \underline{\omega}\times\underline{\rho}_c]^2 dm$$

$$= \frac{1}{2}\int_M [2\underline{v}_c \cdot (\underline{\omega}\times\underline{\rho}_c) + |\underline{v}_c|^2 + (\underline{\omega}\times\underline{\rho}_c)\cdot(\underline{\omega}\times\underline{\rho}_c)]\, dm$$

$$= \frac{1}{2} M |\underline{v}_c|^2 + \underline{v}_c \cdot (\underline{\omega} \times (\overset{0}{\int \underline{\rho}_c dm})) + \frac{1}{2}\int_M (\underline{\omega}\times\underline{\rho}_c)\cdot(\underline{\omega}\times\underline{\rho}_c)\, dm$$

$(\underline{a}\times\underline{b})\cdot\underline{c} = (\underline{a}\ \underline{b}\ \underline{c})$



Note: $\underset{a}{\underline{\omega}} \times \underset{b}{\underline{\rho}_c} \cdot \underset{c}{(\underline{\omega}\times\underline{\rho}_c)} =$ $(\underline{b}\times\underline{c})\cdot\underline{a}$
$\underline{\rho}_c \times (\underline{\omega}\times\underline{\rho}_c)\cdot\underline{\omega}$

$$T = \frac{1}{2} M |\underline{v}_c|^2 + \frac{1}{2}(\underline{\underline{I}}_c \underline{\omega})\cdot\underline{\omega}$$

$$\boxed{T = \underbrace{\frac{1}{2} M |\underline{v}_c|^2}_{\text{translational part}} + \underbrace{\frac{1}{2} \underline{\omega}^T \underline{\underline{I}}_c \underline{\omega}}_{\text{Rotational part}}}$$

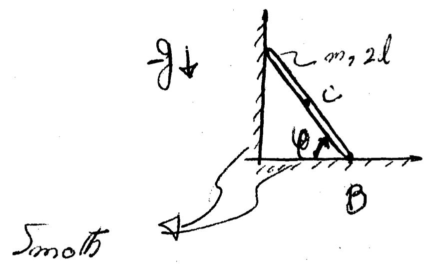
$(\underline{\underline{A}}\,\underline{\underline{B}})^T = \underline{\underline{B}}^T \underline{\underline{A}}^T$

Assume that $\underline{v}_c = 0$ (CM is fixed) then $T > 0$ must hold for any $\underline{\omega} \Rightarrow \underline{\underline{I}}_c$ must be positive definite.

Assume not $\Rightarrow$ principal basis $T = \frac{1}{2}(\omega_1^2 I_1 + \omega_2^2 I_2 + \omega_3^3 I_3)$

Final Note: if $v_B = 0 \Rightarrow \boxed{T = \frac{1}{2}\underline{\omega}^T \underline{\underline{I}}_B \underline{\omega}}$

Example 1:



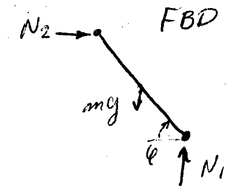
- Equation of Motion?
- Reaction Forces?

$\# DOF = 3 - 2 \times 1 = 1$

$\# DOF$ on Constraint; $\#$ of Constraints

$\Rightarrow$ use 1 generalized Coordinate (φ)

Linear momentum Principle



$$\dot{\underline{P}} = \underline{F}$$
$$= N_2 i + (N_1 - mg) j$$
$$\dot{\underline{P}} = \frac{d}{dt}(m \underline{v}_c) = m \underline{a}_c$$

To Compute a_c; $\quad \underline{v}_c = \underline{v}_B + \underline{\omega} \times \underline{r}_{BC}$

$$= \frac{d}{dt}(2l \cos\varphi)\, \underline{i} + (-\dot{\varphi}\underline{k}) \times (-l\cos\varphi\, i + l \sin(\varphi) j)$$
$$= l\dot{\varphi}(\sin\varphi\, \underline{i} + \cos\varphi\, \underline{j})$$

$$\Rightarrow \underline{a}_c = (l\dot{\varphi}^2 \cos\varphi + l\ddot{\varphi}\sin\varphi)\, \underline{i} + (-l\dot{\varphi}^2 \sin\varphi + l\ddot{\varphi}\cos\varphi)\, \underline{j}$$

(1) $\quad m l \ddot{\varphi} \sin\varphi + m l \dot{\varphi}^2 \cos\varphi = N_2$

(2) $\quad m l \ddot{\varphi} \cos\varphi - m l \dot{\varphi}^2 \sin\varphi = N_1 - mg$

Angular momentum Principle w.r.t. B

$$\dot{\underline{H}}_B + \underline{V}_B \times \underline{P} = \underline{M}_B = (mgl\cos\varphi - N_2\, 2l\sin\varphi)\, \hat{k}$$

$$\underline{H}_B = \underline{H}_C + \underline{P} \times \underline{r}_{CB}$$
$$= \underline{I}_C\, \underline{\omega} + \underline{P} \times \underline{r}_{CB}$$
$$= \frac{1}{12} m (2l)^2 (-\dot{\varphi})\, \underline{k} - m(l\dot{\varphi}\sin\varphi\, i + l\dot{\varphi}\cos\varphi\, j) \times (l\cos\varphi\, \underline{i} - l \sin\varphi\, \underline{j})$$
$$\Rightarrow H_B = \frac{4}{3} m l^2 \dot{\varphi}\, \underline{k}$$

(3) $\quad \Rightarrow \dot{\underline{H}}_B + \underline{V}_B \times \underline{P} = \left(\frac{4}{3} m l^2 \ddot{\varphi} - 2ml^2 \dot{\varphi}^2 \sin\varphi \cos\varphi\right) \underline{k}$

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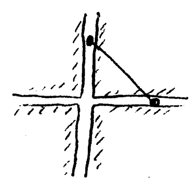
(3): $\frac{4}{3} m l^2 \ddot{\varphi} - 2 m l^2 \dot{\varphi}^2 \sin\varphi \cos\varphi = m g l \cos\varphi - N_2 2 l \sin\varphi$

(1), (3) $\rightarrow$ $\boxed{m l^2 \left(\frac{4}{3} + 2 \sin^2\varphi\right) \ddot{\varphi} - m g l \cos\varphi = 0}$

Eq of motion



Since energy is Conserved (active force in potential, Constraints forces do not work)

$T + V = \text{Const}$

$\frac{d}{dt}(T+V) = 0$ Solve eq of motion

Reaction forces

$N_2 = \frac{3 m g \sin 2\varphi}{4(2 + 3\sin^2\varphi)} + m l \dot{\varphi} \cos\varphi$

Note: $\dot{\varphi}$ Can be expressed as a function of φ from $T + V = \underbrace{T_0 + V_0}_{\text{total Energy}}$

$N_1 = \frac{3 m g \cos^2\varphi}{2(2 + 3\sin^2\varphi)}$

$- m l \dot{\varphi}^2 \sin\varphi + m g$