2.004 Dynamics and Control II
Spring 2008

Massachusetts Institute of Technology

Department of Mechanical Engineering

2.004 *Dynamics and Control II*

Spring Term 2008

Lecture 32[1]

Reading:

- Nise: 10.1
- Class Handout: Sinusoidal Frequency Response

1 Frequency Response and the Pole-Zero Plot (continued)

We showed that if

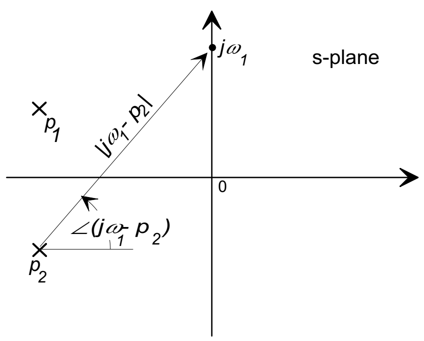
$$H(j\omega) = K\frac{(j\omega - z_1)(j\omega - z_2)\ldots(j\omega - z_{m-1})(j\omega - z_m)}{(j\omega - p_1)(j\omega - p_2)\ldots(j\omega - p_{n-1})(j\omega - p_n)}. \qquad (1)$$

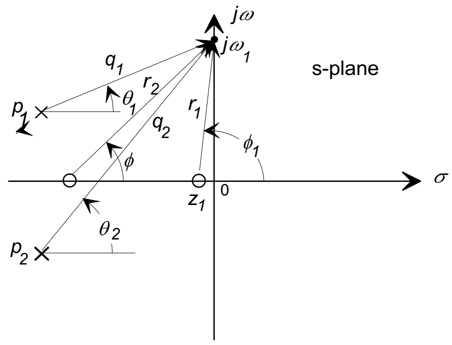
and if each of the vectors from the n system poles to a test point $s = j\omega$ has a magnitude and an angle:

$$\begin{aligned} |j\omega - p_i| &= q_i = \sqrt{\sigma_i^2 + (\omega - \omega_i)^2}, \\ \angle(s - p_i) &= \theta_i = \tan^{-1}\left(\frac{\omega - \omega_i}{-\sigma_i}\right), \end{aligned}$$

and similarly for the m zeros

$$\begin{aligned} |j\omega - z_i| &= r_i = \sqrt{\sigma_i^2 + (\omega - \omega_i)^2}, \\ \angle(s - z_i) &= \phi_i = \tan^{-1}\left(\frac{\omega - \omega_i}{-\sigma_i}\right), \end{aligned}$$





the value of the frequency response at the point $j\omega$ is

$$\begin{aligned} |H(j\omega)| &= K\frac{r_1 \ldots r_m}{q_1 \ldots q_n} \\ \angle H(j\omega) &= (\phi_1 + \ldots + \phi_m) - (\theta_1 + \ldots + \theta_n) \end{aligned}$$

1.0.1 High Frequency Response

In Lecture 31 we saw that at high frequencies all vectors have approximately the same length, that is and

$$\lim_{\omega\to\infty} |H(j\omega)| = K\frac{1}{\omega^{n-m}}$$

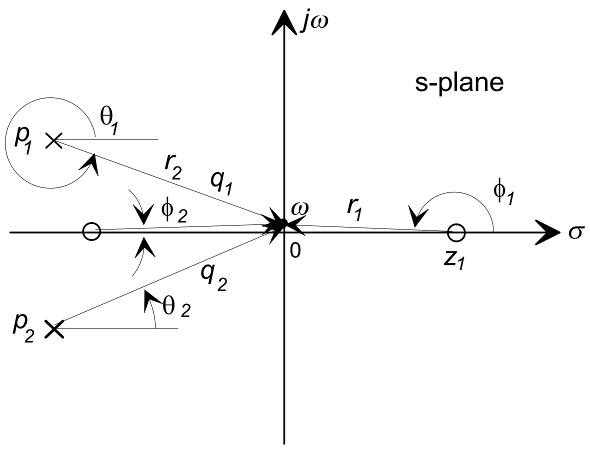
and that all of the angles of the vectors approach $\pi/2$, with the result

$$\lim_{\omega\to\infty} \angle H(j\omega) = -(n-m)\frac{\pi}{2}$$

If a system has an excess of poles over the number of zeros ($n > m$) the magnitude of the frequency response tends to zero as the frequency becomes large. Similarly, if a system has an excess of zeros the gain increases without bound as the frequency of the input increases. If $n = m$ the magnitude function tends to a constant K.

1.0.2 Low Frequency Response

As $\omega \to 0$ we note the following



Magnitude Response: The magnitude response for the s-plane is

$$|H(j\omega)| = K\frac{r_1 \dots r_m}{q_1 \dots q_n}$$

If any of the $r_i \to 0$, then $|H(j\omega)| \to 0$, and if any $q_i \to 0$, then $|H(j\omega)| \to \infty$

If a system has one or more zeros at the origin of the s-plane (corresponding to a pure differentiation), then the system will have zero gain at $\omega = 0$. Similarly, if the system has one or more poles at the origin (corresponding to a pure integration term in the transfer function), the system has infinite gain at zero frequency.

$\lim_{\omega\to 0} \lvert H(j\omega)\rvert = 0$	if there are zeros at the origin
$\lim_{\omega\to 0} \lvert H(j\omega)\rvert = \infty$	if there are poles at the origin
$\lim_{\omega\to 0} \lvert H(j\omega)\rvert = K\frac{r_1 \dots r_m}{q_1 \dots q_n}$	otherwise

Phase Response:

$$\angle H(j\omega) = (\phi_1 + \ldots + \phi_m) - (\theta_1 + \ldots + \theta_n)$$

As $\omega \to 0$:

- All real-axis l.h.p. poles and zeros contribute 0 rad. to the phase response.
- Each complex conjugate pole or zero pair contributes a total of 2π rad. to the phase response (effectively adding 0 rad. to the total response).
- A pole at the origin ($s = 0 + j0$) contributes $-\pi/2$ rad. to the phase response.
- A zero at the origin ($s = 0 + j0$) contributes $+\pi/2$ rad. to the phase response.
- A r.h.p real zero contributes $+\pi$ rad. to the phase response.

The low frequency phase response is therefore

$$\boxed{\lim_{\omega \to 0} \angle H(j\omega) = -(N - M)\frac{\pi}{2} + L\pi \text{ rad.}}$$

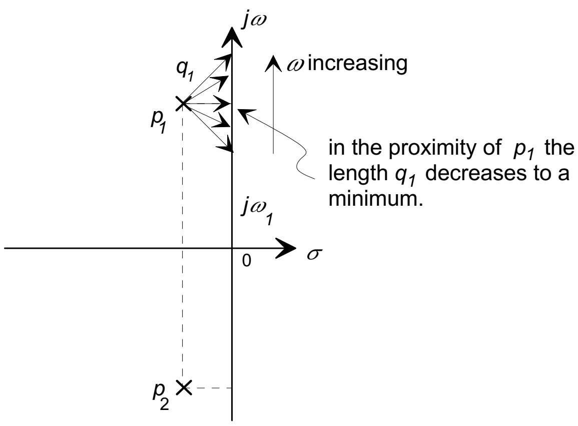
where N is the number of poles at the origin, M is the number of zeros at the origin, and L is the number of r.h.p. real zeros.

1.0.3 Behavior in the Proximity of Poles and Zeros Close to the Imaginary Axis

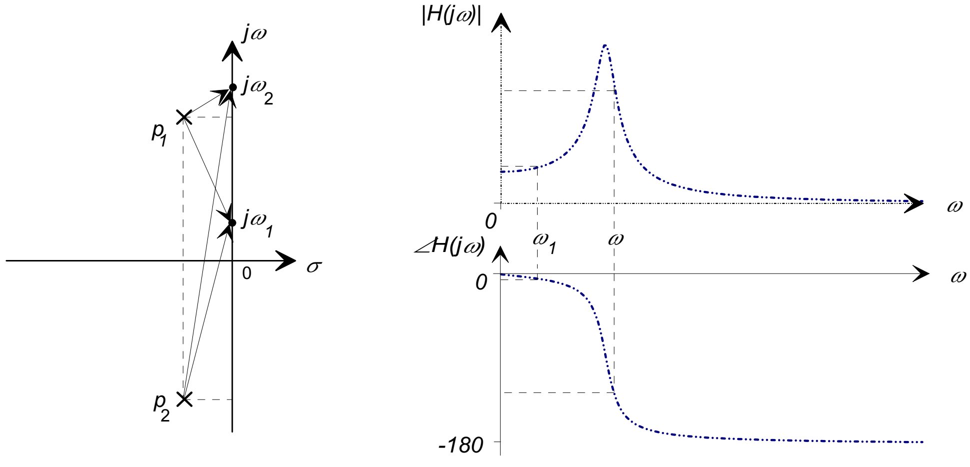
Consider a second-order system with a damping ratio $\zeta \ll 1$, so that the pair of complex conjugate poles are located close to the imaginary axis.

$$\begin{aligned} |H(j\omega)| &= \frac{K}{q_1 q_2} \\ \angle H(j\omega) &= -(\theta_1 + \theta_2) \end{aligned}$$

In this case there are a pair of vectors connecting the two poles to the imaginary axis, and the following conclusions may be drawn by noting how the lengths and angles of the vectors change as the test frequency moves up the imaginary axis: As the input frequency is increased and the test point on the imaginary axis approaches the pole, one of the vectors (associated with the pole in the second quadrant) decreases in length, and at some point reaches a minimum.



Because q_1 appears in the denominator of the magnitude function, over this range there is an *increase* in the value of $|H(j\omega)|$.



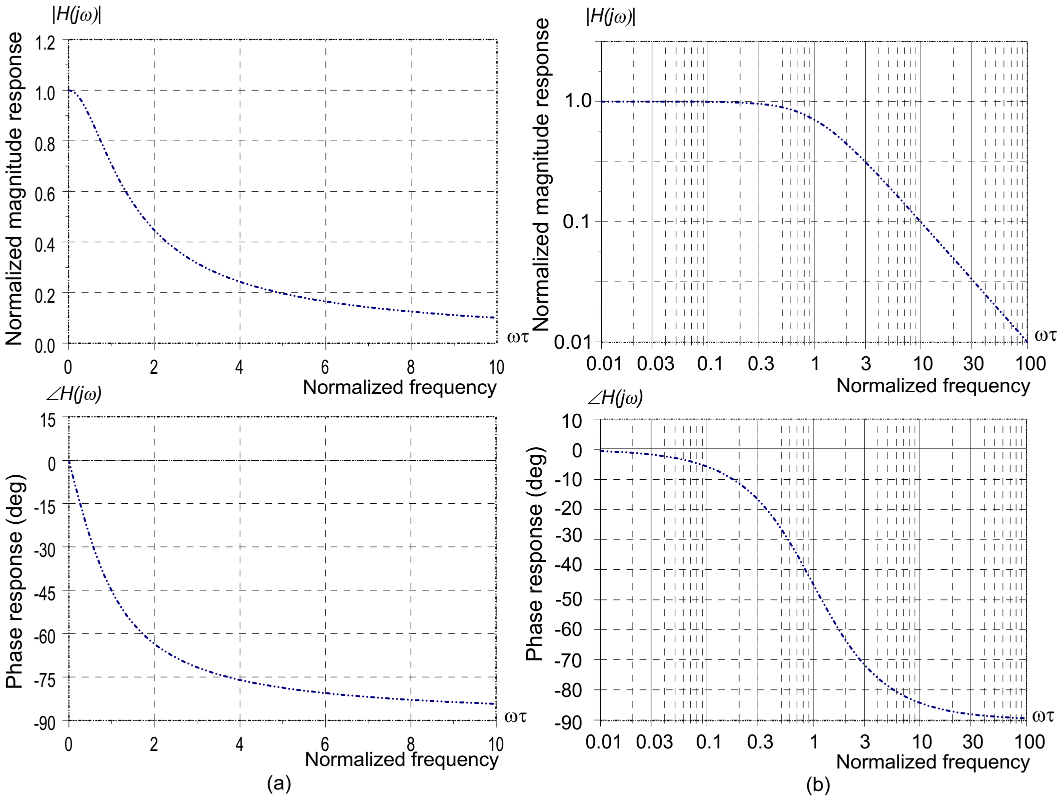
- If a system has a pair of complex conjugate poles close to the imaginary axis, the magnitude of the frequency response has a "peak", or resonance at frequencies in the proximity of the pole. If the pole pair lies directly upon the imaginary axis, the system exhibits an infinite gain at that frequency.

- Similarly, if a system has a pair of complex conjugate zeros close to the imaginary axis, the frequency response. Over this range has a "dip" or "notch" in its magnitude function at frequencies in the vicinity of the zero. Should the pair of zeros lie directly upon the imaginary axis, the response is identically zero at the frequency of the zero, and the system does not respond at all to sinusoidal excitation at that frequency.

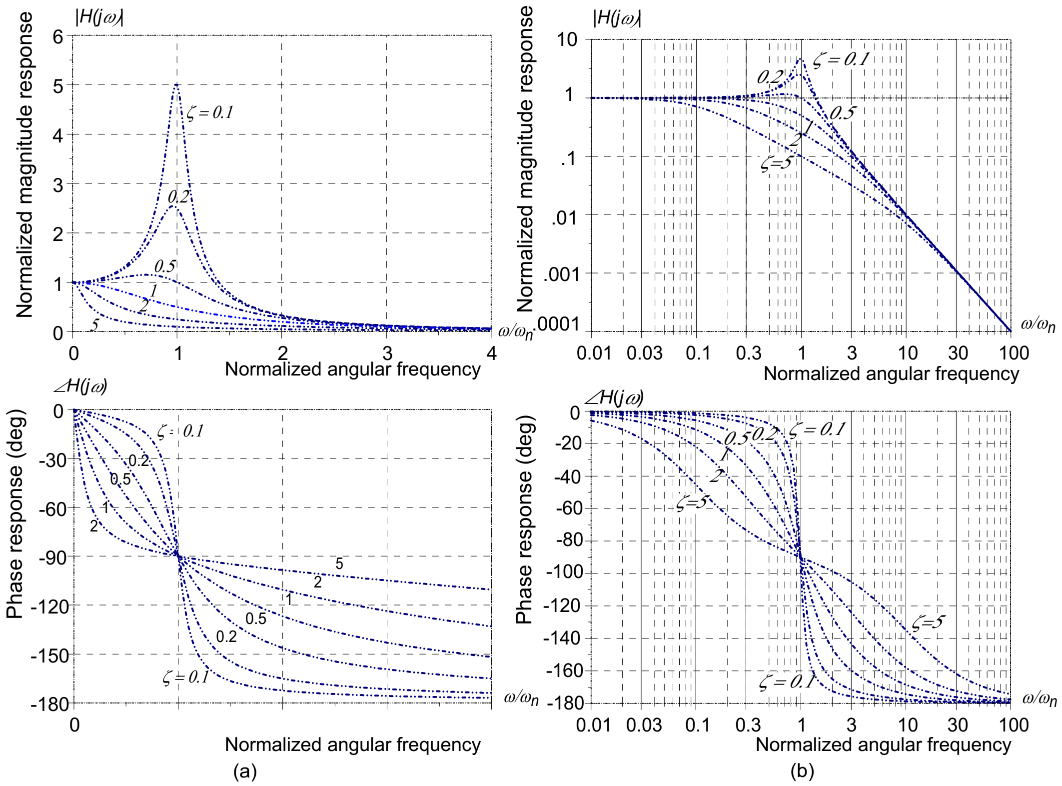
Similarly in the proximity of the pole there is a rapid change of the angle θ_1 associated with the pole p_1.

2 Logarithmic (Bode) Plots

In system dynamic analyses, frequency response characteristics are almost always plotted using logarithmic scales. In particular, the magnitude function $|H(j\omega)|$ is plotted against frequency on a log-log scale, and the phase $\angle H(j\omega)$ is plotted on a linear-log scale. For example, the frequency response functions of a typical first-order system $\tau dy/dt + y = u(t)$ is plotted below on (a) linear axes, and (b) logarithmically scaled axes.



Similarly the second-order frequency response is shown in linear and logarithmic forms below



It can be seen that while two sets of plots convey the same information, they have a different appearance. The logarithmic frequency scale has the effect of expanding the low frequency region of the plots while compressing the high frequencies. The logarithmic magnitude plot can be seen to exhibit straight line asymptotic behavior at high and low frequencies.

In the 1940's H. W. Bode introduced the logarithmic frequency response plots as a simplified method for sketching approximate frequency response characteristics of electronic feedback amplifiers. Bode plots, named after him, have subsequently been widely used in linear system design and analysis, and in feedback control system design and analysis. The Bode sketching method provides an effective means of approximating the frequency response of a complex system by combining of the responses of simple first and second-order systems.

2.1 Logarithmic Amplitude and Frequency Scales:

2.1.1 Logarithmic Amplitude Scale: The Decibel

Bode magnitude plots are frequently plotted using the decibel logarithmic scale to display the function $|H(j\omega)|$. The Bel, named after Alexander Graham Bell, is defined as the logarithm to base 10 of the *ratio* of two *power* levels. In practice the Bel is too large a unit, and the decibel (abbreviated dB), defined to be one tenth of a Bel, has become the standard unit of logarithmic power ratio. The power flow $\mathcal{P}$ into any element in a system, may be expressed in terms of a logarithmic ratio Q to a *reference* power level $\mathcal{P}_{ref}$:

$$Q = \log_{10}\left(\frac{\mathcal{P}}{\mathcal{P}_{ref}}\right) \text{ Bel} \quad \text{or} \quad Q = 10\log_{10}\left(\frac{\mathcal{P}}{\mathcal{P}_{ref}}\right) \text{ dB.} \tag{2}$$

Because the power dissipated in a D–type element is proportional to the square of the amplitude of a system variable applied to it, when the ratio of across or through variables is computed the definition becomes

$$Q = 10\log_{10}\left(\frac{A}{A_{ref}}\right)^2 = 20\log_{10}\left(\frac{A}{A_{ref}}\right) \text{ dB.} \tag{3}$$

where A and A_{ref} are amplitudes of variables.

Note: This definition is only strictly correct when the two amplitude quantities are measured across a common D–type (dissipative) element. Through common usage, however, the decibel has been effectively redefined to be simply a convenient logarithmic measure of amplitude ratio of any two variables. This practice is widespread in texts and references on system dynamics and control system theory.

The table below expresses some commonly used decibel values in terms of the power and amplitude ratios.

Decibels	Power Ratio	Amplitude Ratio
-40	0.0001	0.01
-20	0.01	0.1
-10	0.1	0.3162
-6	0.25	0.5
-3	0.5	0.7071
0	1.0	1.0
3	2.0	1.414
6	4.0	2.0
10	10.0	3.162
20	100.0	10.0
40	10000.0	100.0

The magnitude of the frequency response function $|H(j\omega)|$ is defined as the *ratio* of the amplitude of a sinusoidal output variable to the amplitude of a sinusoidal input variable. This ratio is expressed in decibels, that is

$$20\log_{10}|H(j\omega)| = 20\log_{10}\frac{|Y(j\omega)|}{|U(j\omega)|} \text{ dB}.$$

As noted this usage is not strictly correct because the frequency response function does not define a power ratio, and the decibel is a dimensionless unit whereas $|H(j\omega)|$ may have physical units.

■ Example 1

An amplifier has a gain of 28. Express this gain in decibels.

We note that $28 = 10 \times 2 \times 1.4 \approx 10 \times 2 \times \sqrt{2}$. The gain in dB is therefore $20\log_{10} 10 + 20\log_{10} 2 + 20\log_{10}\sqrt{2}$, or

$$\text{Gain(dB)} = 20 + 6 + 3 = 29 \text{ dB}.$$

The advantages of a logarithmic amplitude scale include:

- Compression of a large dynamic range.
- Cascaded subsections may be handled by addition instead of multiplication, that is

  $$\log(|H_1(j\omega)H_2(j\omega)H_3(j\omega)|) = \log(|H_1(j\omega)|) + \log(|H_2(j\omega)|) + \log(|H_3(j\omega)|)$$

  which is the basis for the sketching rules.
- High and low frequency asymptotes become straight lines when $\log(|H(j\omega)|)$ is plotted against $\log(\omega)$.