13.811
Advanced Structural Dynamics and Acoustics

Lecture 3

The Acoustic Wave Equation

$$\nabla^2\psi - \frac{1}{c^2}\ddot{\psi} = 0$$

$$c = \sqrt{\frac{K}{\rho}}$$

Fourier Transform

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)e^{-i\omega t}d\omega$$

$$f(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$$

$$\nabla^2\psi_\omega + k^2\psi_\omega = 0$$

Acoustic Wavenumber

$$k = \frac{\omega}{c}$$

1-dimensional propagation

$$\frac{\partial}{\partial y} = 0; \frac{\partial}{\partial z} = 0 \Rightarrow \frac{\partial^2\psi_\omega}{\partial x^2} + k^2\psi_\omega = 0$$

Plane Waves

$$\psi_\omega(x) = Ae^{i}kx + Be^{-ikx}$$

$$\psi(t,x) = \int_{-\infty}^{\infty}\left[Ae^{i(kx-\omega t)} + Be^{i(kx+\omega t)}\right]d\omega$$

$f(x - ct)$ — Forward Propagating

$g(x + ct)$ — Backward Propagating

The Acoustic Wave Equation

1 D Plane Wave

$$\psi(x,t) = e^{i(kx-\omega t)}$$

Field Parameters

Particle Displacement $$u_x = \frac{\partial \psi}{\partial x} = ike^{i(kx-\omega t)}$$

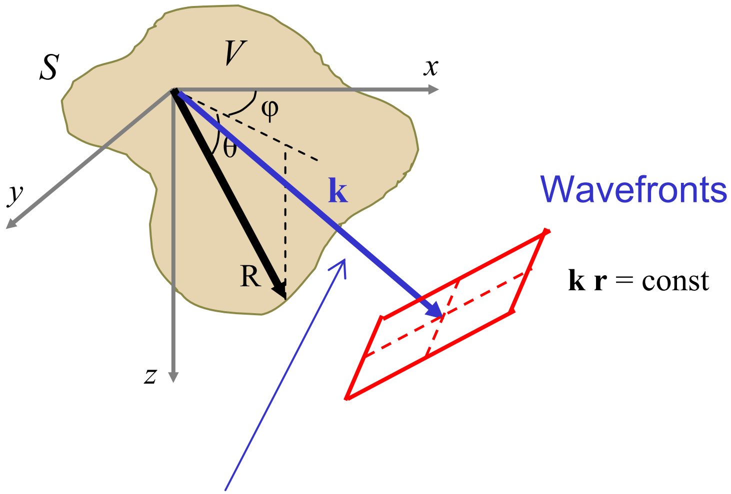
Particle Velocity $$v_x = \frac{\partial u_x}{\partial t} = k\omega e^{i(kx-\omega t)} = \frac{\omega^2}{c} e^{i(kx-\omega t)}$$

Acoustic Pressure $$p = -\rho \frac{\partial^2 \psi}{\partial t^2} = \rho\omega^2 e^{i(kx-\omega t)}$$

$$\Rightarrow p = \boxed{\rho c} v_x = \boxed{Z} v_x$$

Acoustic Impedance (Z)

3D Acoustic Plane Waves



$$\psi_\omega(x,y,z) = Ae^{i\mathbf{k}\cdot\mathbf{r}} = Ae^{i(k_x x + k_y y + k_z z)}$$

Wavevector – Propagation Direction

$$k_x^2 + k_y^2 + k_z^2 = k^2$$

k_x , k_y , k_z not independent

Vertical Wavenumber - Dependent Variable

$$k_z^2 = k^2 - k_x^2 - k_y^2 = k^2 - k_r^2$$

Radial Wavenumber

$$k_r = \sqrt{k_x^2 + k_y^2}$$

2D Fourier Transform

$$f(x,y) = \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} f(k_x,k_y) e^{ik_x x} dk_x] e^{ik_y y} dk_y$$

$$f(k_x,k_y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} f(x,y) e^{-ik_x x} dx] e^{-ik_y y} dy$$

Depth-separated Helmholtz Equation

$$\left[\frac{d^2}{dz^2} + \left(k^2 - k_x^2 - k_y^2\right)\right] \psi_\omega(k_x,k_y;z) = 0$$

$$\psi_\omega(k_x,k_y;z) = A(k_x,k_y)e^{ik_z z} + B(k_x,k_y)e^{-ik_z z}$$

$$k_x, k_y \in [-\infty, \infty]$$

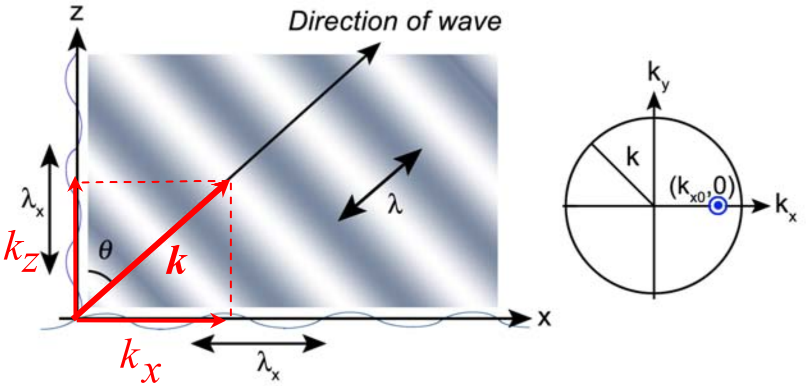
Vertical Wavenumber

$$k_z = \pm\sqrt{k^2 - k_r^2}$$

$$k_z = \pm\sqrt{k^2 - k_r^2} = \pm i\sqrt{k_r^2 - k^2}$$

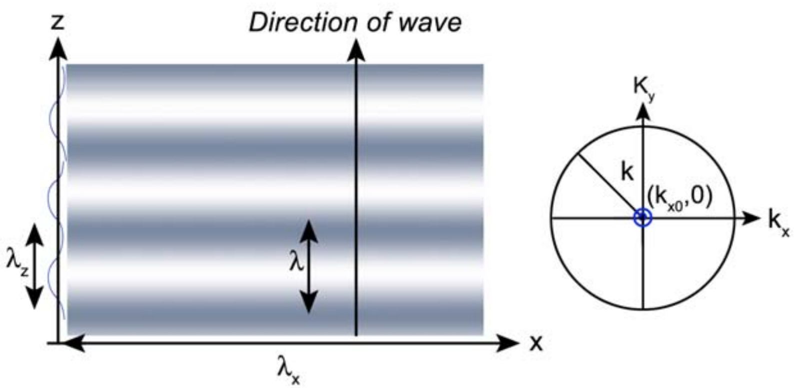
3D Acoustic Plane Waves

$$k_y = 0 \qquad |k_x| < k$$



$$k_x = k \sin\theta$$

$$k_z = k \cos\theta$$



2D Fourier Transform

$$f(x,y) = \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} f(k_x,k_y) e^{ik_x x} dk_x] e^{ik_y y} dk_y$$

$$f(k_x,k_y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} f(x,y) e^{-ik_x x} dx] e^{-ik_y y} dy$$

Depth-separated Helmholtz Equation

$$\left[\frac{d^2}{dz^2} + \left(k^2 - k_x^2 - k_y^2\right)\right] \psi_\omega(k_x,k_y;z) = 0$$

Radiation Condition

$$\psi_\omega(k_x,k_y;z) = \begin{cases} A(k_x)e^{ik_z z} & |k_x| \le k \\ A(k_x)e^{-\alpha z} & |k_x| \quad k \end{cases}$$

$$= A(k_x)e^{ik_z z}$$

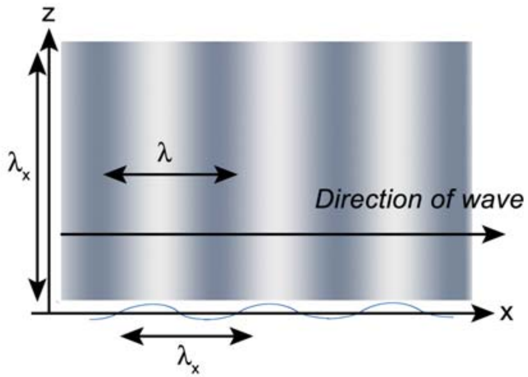
Vertical Wavenumber

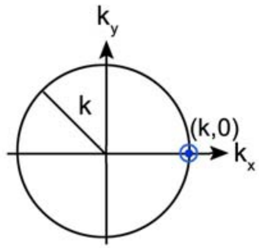
$$k_z = \begin{cases} \sqrt{k^2 - k_r^2} & |k_r| \le k \\ i\sqrt{k_r^2 - k^2} & |k_r| > k \end{cases}$$

3D Acoustic Plane Waves

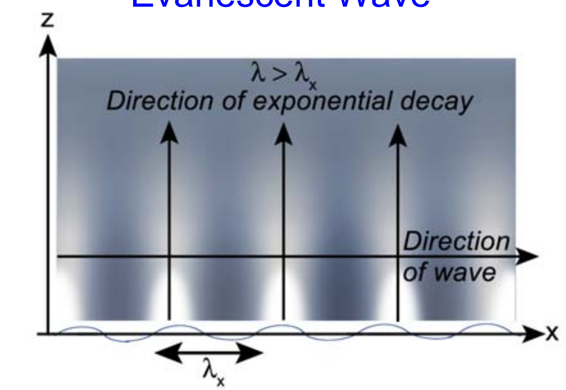
$$k_y = 0 \qquad |k_x| < k$$

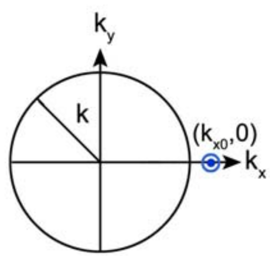
Grazing Plane Wave





Evanescent Wave





2D Fourier Transform

$$f(x,y) = \int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} f(k_x,k_y)e^{ik_x x}dk_x]e^{ik_y y}dk_y$$

$$f(k_x,k_y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} f(x,y)e^{-ik_x x}dx]e^{-ik_y y}dy$$

Depth-separated Helmholtz Equation

$$\left[\frac{d^2}{dz^2} + \left(k^2 - k_x^2 - k_y^2\right)\right]\psi_\omega(k_x,k_y;z) = 0$$

Radiation Condition

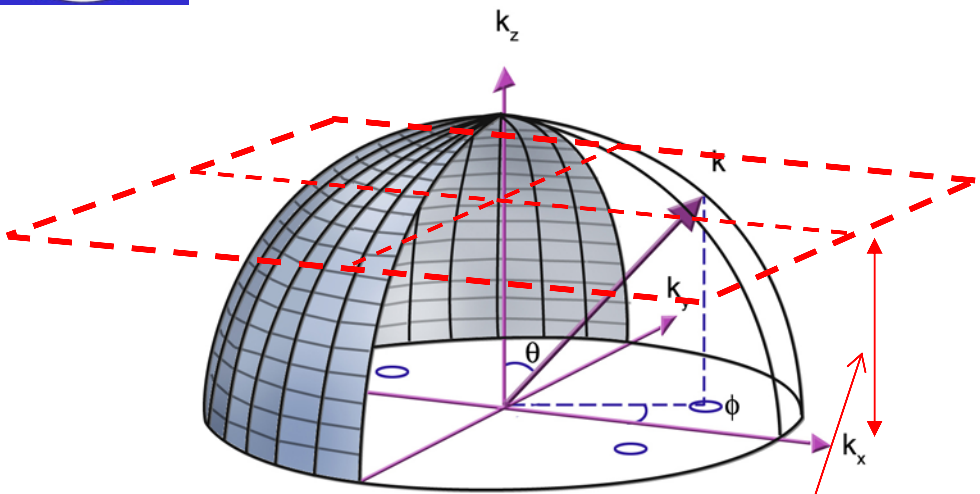
$$\psi_\omega(k_x,k_y;z) = \begin{cases} A(k_x)e^{ik_z z} & |k_x| \le k \\ A(k_x)e^{-\alpha z} & |k_x| > k \end{cases}$$

$$= A(k_x)e^{ik_z z}$$

Vertical Wavenumber

$$k_z = \begin{cases} \sqrt{k^2 - k_r^2} & |k_r| \le k \\ i\sqrt{k_r^2 - k^2} & |k_r| > k \end{cases}$$

3D Acoustic Plane Waves



Wavefield Extrapolation

$$\psi_\omega(k_x, k_y; z) = \begin{cases} A(k_x)e^{ik_z z} & |k_x| \le k \\ A(k_x)e^{-\alpha z} & |k_x| < k \end{cases}$$

$$= A(k_x)e^{ik_z z}$$

Vertical Wavenumber

$$k_z = \begin{cases} \sqrt{k^2 - k_r^2} & |k_r| \le k \\ i\sqrt{k_r^2 - k^2} & |k_r| > k \end{cases}$$

Radial Wavenumber

$$k_r^2 = k_x^2 + k_y^2$$

$$k_z^2 = k^2 - k_r^2$$

Radiating Waves $|k_r| \le k$

Evanescent Waves $|k_r| > k$

Propagation Angles

$$k_r = k\sin\theta$$

$$k_x = k\sin\theta cos\phi$$

$$k_y = k\sin\theta sin\phi$$

Wavefield Extrapolation

At each radial wavenumber the field at any other depth z is determined by multiplication by a simple exponential

Radiation From Infinite Plate

Depth-separated Solution

$$\psi_\omega(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_\omega(k_x,k_y;z)e^{ik_x x}e^{ik_y y}dk_x dk_y$$

$$\psi_\omega(k_x;z) = A(k_x)e^{ik_z z}$$

Vertical Displacement $$w_\omega(k_x;z) = \frac{\partial \psi_\omega}{\partial z} = ik_z A(k_x)e^{ik_z z}$$

Image removed due to copyright considerations.
See Figure 2.10 in Williams, E. G. *Fourier Acoustics*.
London: Academic Press, 1999

Boundary Condition

$$w_\omega(k_x;z)|_{z=0} \quad = \quad w_\omega(k_x,0)$$

$$\Leftrightarrow$$

$$\int_{-\infty}^{\infty} \boxed{ik_z A(k_x)} e^{ik_x x}dk_x \quad = \quad \frac{e^{ik_{x0}x}+e^{-ik_{x0}x}}{2}$$

$$ik_z A(k_x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ik_{x0}x}+e^{-ik_{x0}x}}{2}e^{-ik_x x}dx$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik_x x}e^{-iq_x x}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k_x-q_x)x}dx = \delta(k_x-q_x)$$

$$ik_z A(k_x) \quad = \quad \frac{\delta(k_x-k_{x0})+\delta(k_x+k_{x0})}{2}$$

$$\Leftrightarrow$$

$$\boxed{A(k_x) \quad = \quad -\frac{i}{2k_z}\delta(k_x \pm k_{x0})}$$

Plate Vibration

$$k_y = 0$$

$$w_\omega(x,0) = cosk_{x0}x$$

$$\lambda_x = \frac{2\pi}{k_{x0}}$$

Radiation From Infinite Plate

Radiation Regime

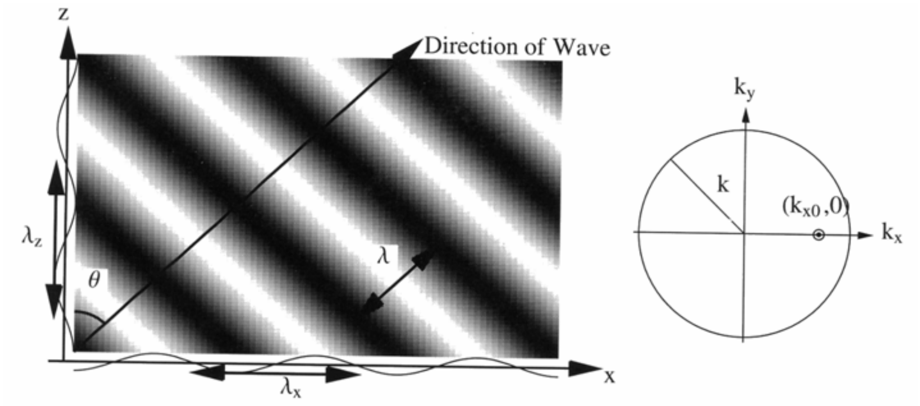
Plate Vibration

$$k_y = 0$$

$$w_\omega(x,0) = cos k_{x0} x$$

$$\lambda_x = \frac{2\pi}{k_{x0}}$$

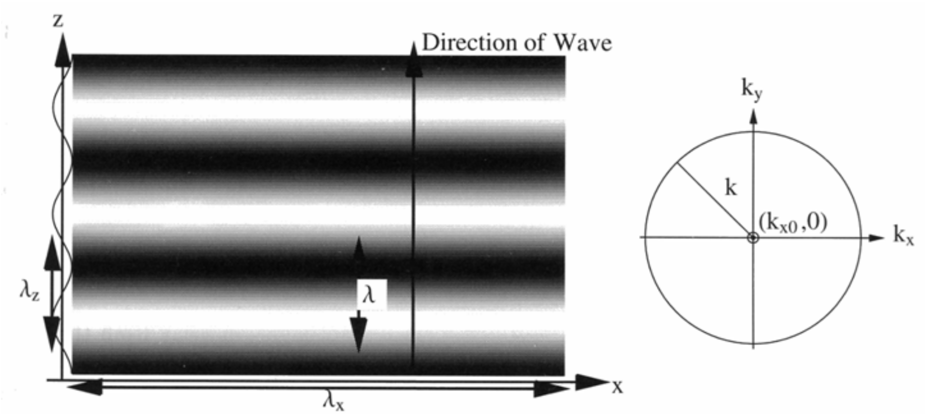
$$\delta(k_x - k_{x0})$$



Radiated Field

$$\psi_\omega(k_x; z) = A(k_x) e^{ik_z z}$$

$$A(k_x) = -\frac{i}{2k_z}\delta(k_x \pm k_{x0})$$



$$k_y = 0 \qquad |k_x| < k$$

Radiation From Infinite Plate

Evanescent Regime

Plate Vibration

$$k_y = 0$$

$$w_\omega(x,0) = cos k_{x0} x$$

$$\lambda_x = \frac{2\pi}{k_{x0}}$$

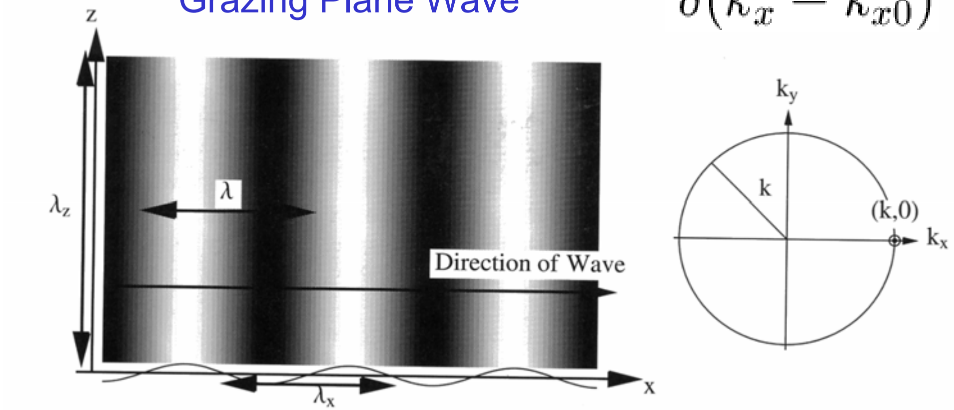
Evanescent Field

$$\psi_\omega(k_x; z) = A(k_x) e^{ik_z z}$$

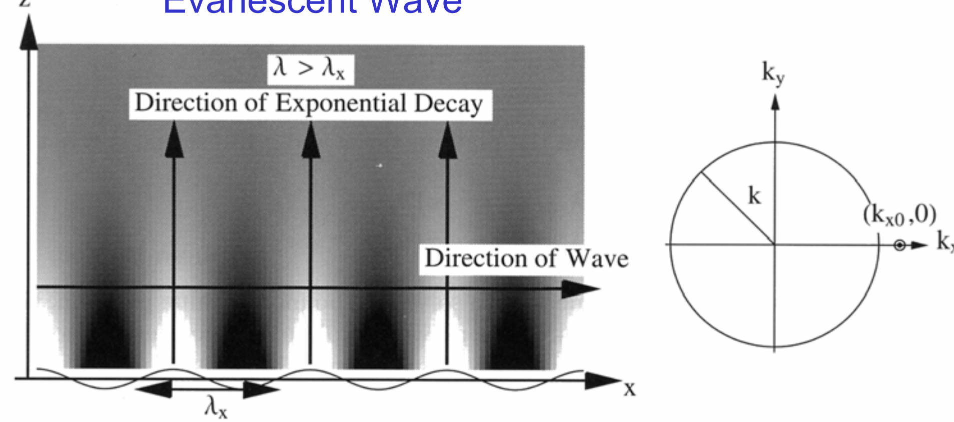
$$A(k_x) = -\frac{i}{2k_z}\delta(k_x \pm k_{x0})$$

Grazing Plane Wave

$$\delta(k_x - k_{x0})$$



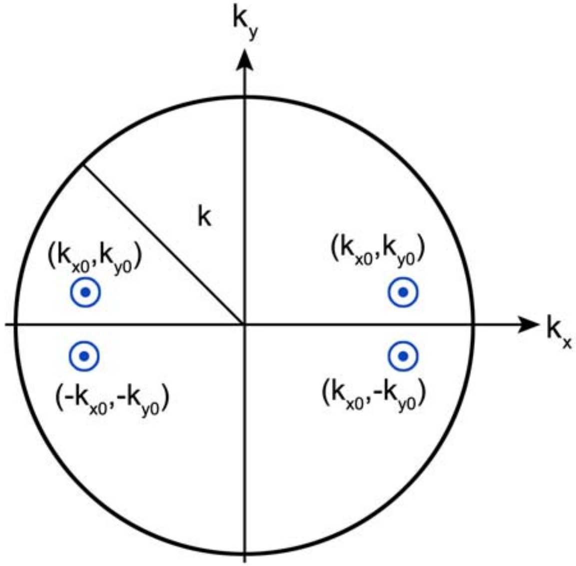
Evanescent Wave

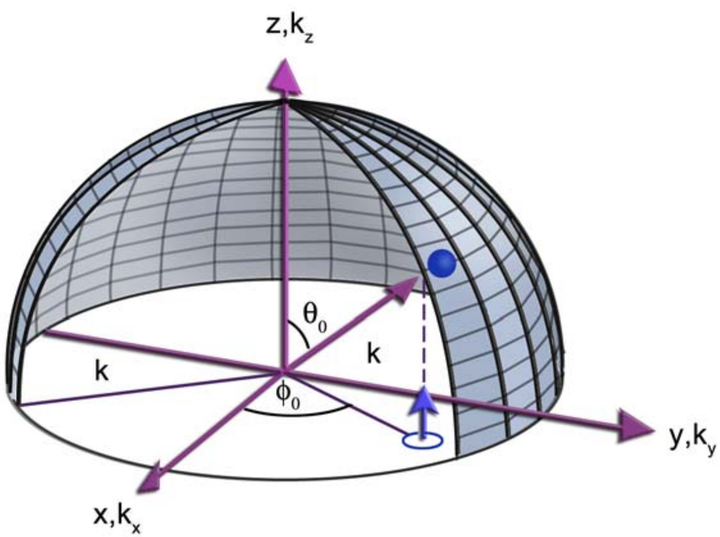


$$k_y = 0 \qquad |k_x| > k$$

3D Radiation From Infinite Plate

Radiation Regime





δ(k_x-k_xo)δ(k_y-k_yo)

$$\begin{aligned} w_\omega(x,y;0) &= \cos k_{x0}x \cos k_{y0}y \\ &= (e^{ik_{x0}x} + e^{-ik_{x0}x})(e^{ik_{y0}y} + e^{-ik_{y0}y})/4 \\ &= \left[e^{ik_{x0}x}e^{ik_{y0}y} + e^{-ik_{x0}x}e^{ik_{y0}y} + e^{ik_{x0}x}e^{-ik_{y0}y} + e^{-ik_{x0}x}e^{-ik_{y0}y}\right]/4 \end{aligned}$$

$$A(k_x,k_y) = -\frac{i}{4k_z}\left[\delta(k_x \pm k_{x0})\delta(k_y \pm k_{y0})\right]$$

3D Radiation From Infinite Plate

Evanescent Regime

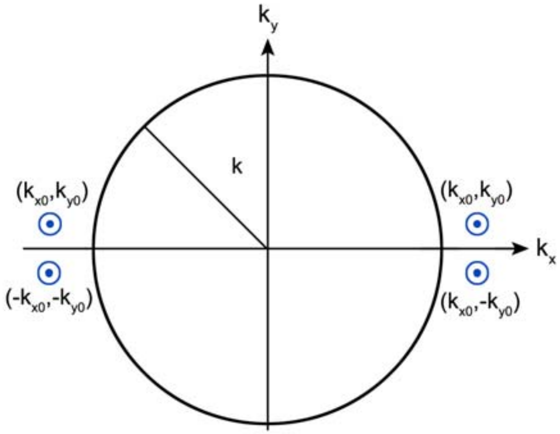


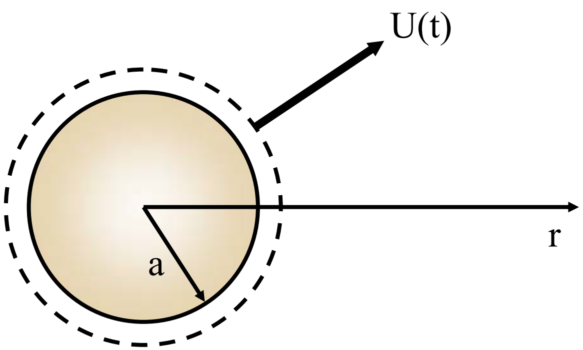
Image removed due to copyright considerations.
See Figure 2.10 in [Williams].

$$\lambda_x = \frac{2\pi}{k_{x0}} \qquad \lambda_y = \frac{2\pi}{k_{y0}}$$

$$\begin{aligned} w_\omega(x,y;0) &= \cos k_{x0}x \cos k_{y0}y \\ &= (e^{ik_{x0}x} + e^{-ik_{x0}x})(e^{ik_{y0}y} + e^{-ik_{y0}y})/4 \\ &= \left[e^{ik_{x0}x}e^{ik_{y0}y} + e^{-ik_{x0}x}e^{ik_{y0}y} + e^{ik_{x0}x}e^{-ik_{y0}y} + e^{-ik_{x0}x}e^{-ik_{y0}y}\right]/4 \end{aligned}$$

$$A(k_x,k_y) = -\frac{i}{4k_z}\left[\delta(k_x \pm k_{x0})\delta(k_y \pm k_{y0})\right]$$

Radiation of Sound
The Point Source



Unbounded Homogeneous Medium

Frequency Domain

$$u_r(a) = U(\omega) \, .$$

Spherical geometry solution

$$\psi(r) = A \frac{e^{ikr}}{r} ,$$

$$u_r(r) = \frac{\partial \psi(r)}{\partial r} = A \, e^{ikr} \left(\frac{ik}{r} - \frac{1}{r^2} \right) .$$

Simple Point Source

$$ka \ll 1$$

$$u_r(a) = A \, e^{ika} \frac{ika - 1}{a^2} \simeq -\frac{A}{a^2} ,$$

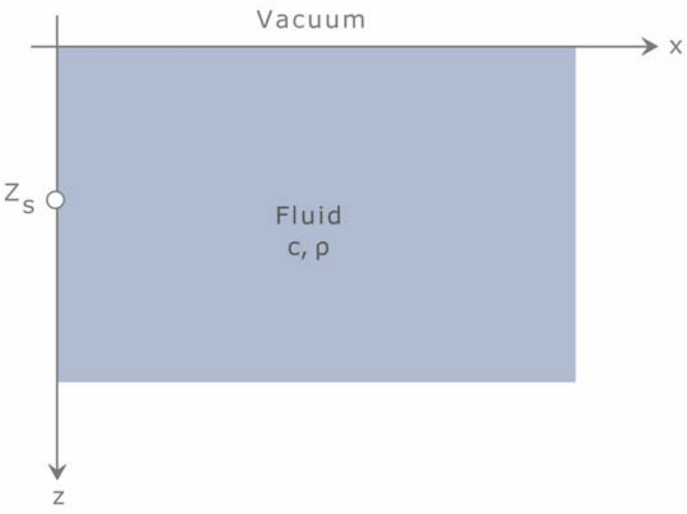
$$A = -a^2 \, U(\omega) \, .$$

$$\Rightarrow$$

$$\psi(r) = -S_\omega \frac{e^{ikr}}{4\pi r} .$$

$$S_\omega = 4\pi a^2 \, U(\omega)$$

Point Source in Halfspace



Homogeneous Solution

$$H_\omega(k_r, z) = A^+(k_r)\, e^{ik_z z} + A^-(k_r)\, e^{-ik_z z}\,,$$

Vertical Wavenumber

$$\begin{aligned} k_z &= \sqrt{k^2 - k_r^2} \\ &= \begin{cases} \sqrt{k^2 - k_r^2}\,, & k_r \le k \\ i\sqrt{k_r^2 - k^2}\,, & k_r > k\,. \end{cases} \end{aligned}$$

Radiation Conditions

$$H_\omega(k_r, z) = \begin{cases} A^+(k_r)\, e^{ik_z z}\,, & z \to +\infty \\ A^-(k_r)\, e^{-ik_z z}\,, & z \to -\infty\,. \end{cases}$$

Point Source in Halfspace

$$g_\omega(k_r, z, z_s) = A(k_r) \begin{cases} e^{ik_z(z-z_s)}, & z \geq z_s \\ e^{-ik_z(z-z_s)}, & z \leq z_s \end{cases}$$

$$= A(k_r)\, e^{ik_z|z-z_s|}\,.$$

Integration of depth-separated wave equation over $[z_s - \epsilon, z_s + \epsilon]$:

$$\left[\frac{dg_\omega(k_r, z)}{dz}\right]_{z_s-\epsilon}^{z_s+\epsilon} + O(\epsilon) = -\frac{1}{2\pi}\,.$$

$$\Rightarrow$$

$$A(k_r) = -\frac{1}{4\pi i k_z}$$

$$\Rightarrow$$

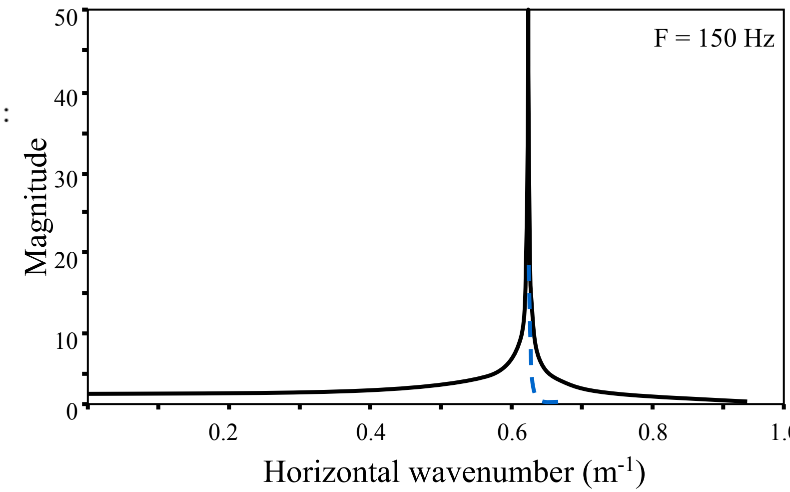
$$g_\omega(k_r, z, z_s) = -\frac{e^{ik_z|z-z_s|}}{4\pi i k_z}\,.$$

Inverse Hankel Transform - Sommerfeld–Weyl Integral

$$g_\omega(r, z, z_s) = \frac{i}{4\pi} \int_0^\infty \frac{e^{ik_z|z-z_s|}}{k_z} J_0(k_r r)\, k_r\, dk_r\,,$$

Grazing Angle Representation

$$k_x = k\cos\theta\,,$$
$$k_z = k\sin\theta\,,$$
$$\frac{dk_x}{d\theta} = -k_z\,.$$



$$\Rightarrow$$

$$g_\omega(\mathbf{r}, \mathbf{r}') \simeq \frac{i}{4\pi} \int_{-k}^{k} \frac{e^{ik_z|z-z_s|}}{k_z} e^{ik_x x}\, dk_x$$

$$= \frac{i}{4\pi} \int_0^\pi e^{ik\,|z-z_s|\sin\theta + ikx\cos\theta}\, d\theta\,.$$

Point Source in Halfspace

$$\psi(k_r, 0) \equiv 0$$

$$\psi(k_r, z) \text{ radiating for } z \to \infty$$

$$\Rightarrow$$

$$\begin{aligned}\psi(k_r, 0) &= -S_\omega \left[g_\omega(k_r, 0, z_s) + H_\omega(k_r, 0)\right] \\ &= S_\omega \left[\frac{e^{ik_z z_s}}{4\pi i k_z} - A^+(k_r)\right] = 0\,,\end{aligned}$$

Total field

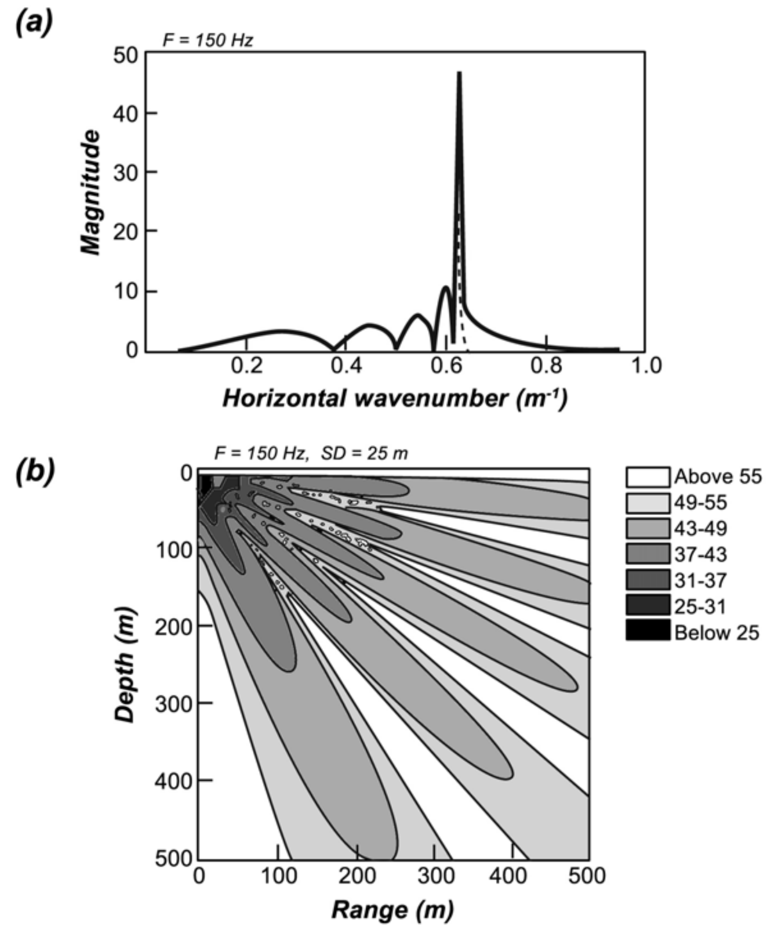
$$\psi(k_r, z) = S_\omega \left[\frac{e^{ik_z|z-z_s|}}{4\pi i k_z} - \frac{e^{ik_z(z+z_s)}}{4\pi i k_z}\right].$$

Loyd-Mirror Minima and Maxima

$$\begin{aligned}\sin\theta_{\max} &= \frac{(2m-1)\,\pi}{2kz_s}\,, \\ \sin\theta_{\min} &= \frac{(m-1)\,\pi}{kz_s}\,.\end{aligned}$$

Free Surface Reflection Coefficient

$$R = -1\,.$$



Source in a fluid halfspace.
(a) Magnitude of the depth-dependent Green's function.
Solid curve: $z-z_y = \lambda/10$. Dashed curve: $z-z_x=2\lambda$
(b) Pressure field contours