Ain Sonin and Gareth McKinley — **Solutions to Quiz 2** — 2.25 Fall 2004

Problem 1 The transition from dripping to jetting

See C. Clanet & J. Lasheras, "Transition from Dripping to Jetting", *J. Fluid Mech* **383**, 1999 p307 for more details on this problem!

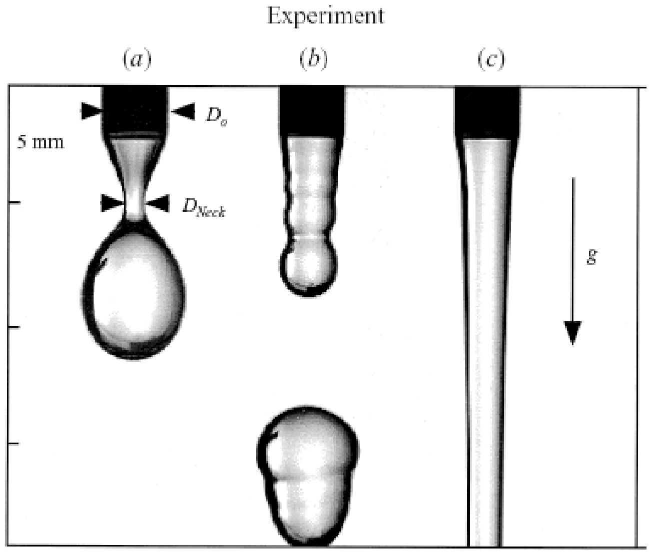


Figure 1

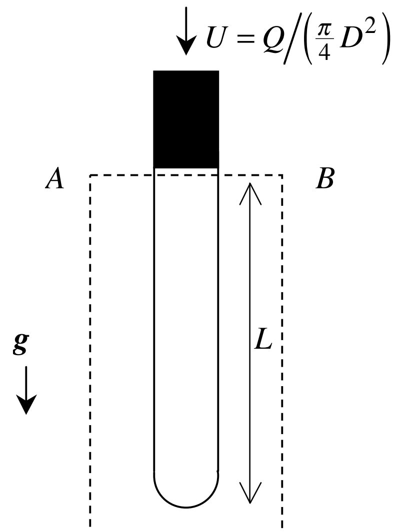


Figure 2 (snapshot at a single instant in time *t*)

For fluids exiting from typical size orifices, viscous stresses are negligible because a dimensionless parameter known as the *Ohnesorge number* is small. This is defined as $Oh = \mu\big/\sqrt{\rho\sigma D}$. For the case here we thus have $Oh \approx 10^{-3}\big/\sqrt{10^3(0.07)(0.005)} \approx 0.0017$!

(a) The surface is cylindrical and there is no tangential stress (no viscous effects). Normal to the jet we have pressure and surface tension acting. The principal radii of curvature for a cylinder are such that the mean curvature is given by:

$$2H = \frac{1}{R_a} + \frac{1}{R_b} = \frac{1}{(D/2)} + \frac{1}{\infty} = \frac{2}{D} \tag{1}$$

$$\text{hence } p_i\mathbf{n} = \left(p_a + 2H\sigma\right)\mathbf{n} \quad \rightarrow p_i = p_a + \frac{2\sigma}{D} \tag{2}$$

The <u>additional</u> axial force arising from surface tension acting along the axial direction of the jet is $F_z = \pi D\sigma$ (remember surface tension is a line force – proportional to length).

(b) Criterion: *The flux of momentum into the control volume shown must always be greater than zero* (otherwise a stationary pendant drop will have formed which is attached to the orifice).

The 'A form' of the conservation of linear momentum is most appropriate; this gives

$$\frac{d\mathbf{P}_{cv}}{dt} + \int_{CS(t)} \rho v(v - v_c)\cdot\mathbf{n}\, dA = \sum \mathbf{F}_{cv} \tag{3}$$

For the *z-component* of linear momentum with positive *z* in direction of gravity we thus find:

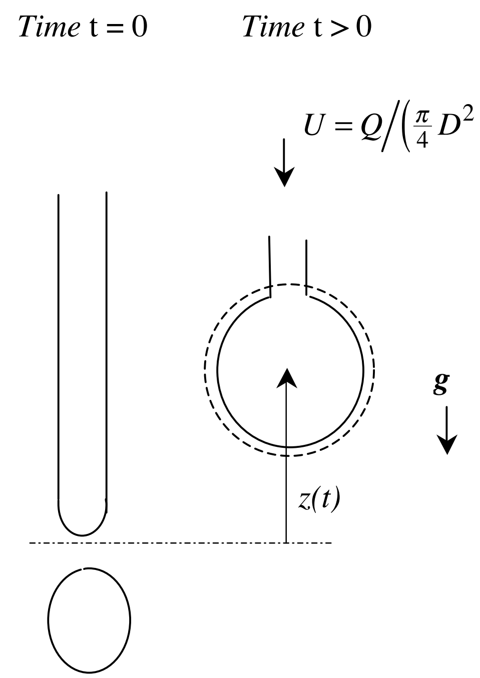
$$\frac{dP_{cv}}{dt} - \rho U^2 \tfrac{1}{4}\pi D^2 + 0 = -\pi D\sigma - p_a(\tfrac{1}{4}\pi D^2) + \left(p_a + \frac{2\sigma}{D}\right)(\tfrac{1}{4}\pi D^2) + (\tfrac{1}{4}\pi D^2)\rho g L \qquad (4)$$

(NB there is an inflow but no outflow flux term – and don't forget atmospheric pressure pushing upwards on bottom of the fluid column). On rearranging and simplifying, one finds

$$U > \sqrt{\left(\frac{2\sigma}{\rho D} - gL\right)} \qquad (5)$$

in dimensionless form $\frac{\rho U^2 D}{\sigma} > 2\left\{1 - \frac{\rho g L D}{2\sigma}\right\}$ where the Weber number is $We = \rho U^2 D/\sigma$.

(c) Unsteady momentum balance as shown in the figure:
NB here we consider a coordinate system oriented vertically **upwards** (so that $\mathbf{g} = -g\boldsymbol{\delta}_z$). We ignore the small contribution of the mass in the long thin jet and consider only the large terminal drop which has a mass $M_{cv}(t)$ and a velocity $\dot{z}(t)$.



Conservation of mass gives:

$$\frac{dM_{cv}}{dt} - \rho(U + \dot{z})A = 0, \qquad (6)$$

where for convenience we write $A = \frac{1}{4}\pi D^2$ henceforth.
Conservation of linear momentum for the control volume shown in the figure gives:

$$\frac{d(M_{cv}(t)\dot{z})}{dt} + (-\rho U)\big(-(U+\dot{z})\big)A + 0 = \pi D\sigma - \frac{\pi D}{2}\sigma - M_{cv}(t)g, \quad (7)$$

where the incoming linear momentum (per unit volume) is $-\rho U\boldsymbol{\delta}_z$.

TABLE OF IMPORTANT TERMS

Term	Inflow	Outflow
Outward facing normal $\boldsymbol{n}$	$+\boldsymbol{\delta}_z$	$-\boldsymbol{\delta}_z$
Velocity vector $\boldsymbol{v}$	$-U\boldsymbol{\delta}_z$	0
Control surface velocity, $\boldsymbol{v_c}$	$+\dot{z}\boldsymbol{\delta}_z$	0
Normal component of velocity relative to CS.	$v_{rn} = (\boldsymbol{v} - \boldsymbol{v_c})\cdot\mathbf{n} = -(U+\dot{z})$	0

(d) With boundary condition $z(t = 0) = 0$ integrating equation (6) for mass gives

$$M_{cv} = \rho A(z + Ut) \quad \text{(i.e. increases linearly with both } z \text{ and time)}$$

Expanding (2) gives $$\frac{dM_{cv}}{dt}\dot{z} + M_{cv}\ddot{z} + \rho UA(\dot{z}+U) = \tfrac{1}{2}\pi D\sigma - M_{cv}g\,. \qquad (8)$$

Eliminating the mass $M_{cv}(t)$ from this expanded form of equation (7) gives following nonlinear second order differential equation for the position $z(t)$:

$$\dot{z}(U+\dot{z}) + (z+Ut)\ddot{z} + U(\dot{z}+U) = \frac{\pi D\sigma}{2\rho A} - g(z+Ut)\,. \qquad (9)$$

Substituting the simple quadratic forms $z(t) = \tfrac{1}{2}at^2 + (b-U)t$, $\dot{z} = (b-U) + at$ and $\ddot{z} = a$ gives three equations for terms of order t^2, t^1 and t^0 respectively:

Gathering terms at order t^2 gives $a = -g/3$
Gathering terms at order t^1 gives $a = -g/3$ also (i.e. solution is consistent)

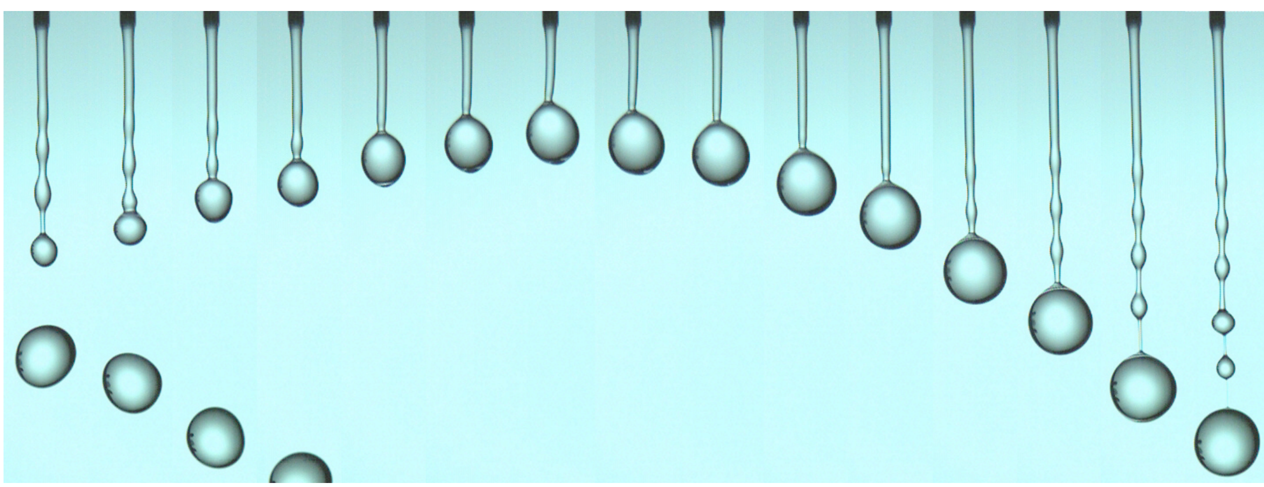
and gathering terms at order t^0 gives $b = \sqrt{\dfrac{\pi D\sigma}{2\rho A}} = \sqrt{\dfrac{2\sigma}{\rho D}}$ (by eliminating area *A*)

<u>The following is not part of the quiz but might be of interest:</u>

this final parameter *b* is the characteristic velocity of an inertio-capillary wave which travels along the surface of the jet. This wave (connected to the pendant drop) travels *upstream* against the downwash of linear momentum of the jet (imagine a salmon swimming upstream!). The trajectory of the pendant drop is thus

$$z = -\tfrac{1}{6}gt^2 + (V_{wave} - U)t \qquad (10)$$

If $U < V_{wave}$ then the initial velocity is *upwards*. Eventually the incoming negative-momentum (downwards) wins and the parabolic trajectory reaches a maximum height of $z_{\max} = 3(V_{wave}-U)^2/(2g)$ and then the drop accelerates downwards until it breaks and the process repeats…



…..conservation of mass and linear momentum in action

Note that the fluid in this case is actually a dilute polymer solution. The polymer serves to stabilize the thin ligaments that connect the pendant drop to the jet; however the elasticity does not make a dominant contribution to the momentum balance.

2.25 SOLUTION, PROB. 2, QUIZ 2, 2004

A. Mass conservation for a fixed CV between x=0 and x=x:

$$\frac{d}{dt}\int_{CV} \rho dV = -\int_{CS} \rho V_n dA \tag{1}$$

Canceling densities and substituting for the volume integral, we get

$$\frac{d}{dt}\left[\pi R^2 x\right] = -Q(x,t) \tag{2}$$

where Q is the local volume flow rate at x,t. The local mean-velocity distribution is

$$u(x,t) = \frac{Q(x,t)}{\pi R^2(t)}. \tag{3}$$

Eliminating Q between (2) and (3), we get the mean-velocity distribution

$$u(x,t) = \frac{2\kappa x}{R(t)} \tag{4}$$

where

$$\kappa = -dR/dt. \tag{5}$$

B. The pressure distribution is obtained by recognizing that in the locally-fully-developed (inertia-free) approximation, the local volume flow rate in a tube is

$$Q = -\frac{\pi R^4}{8\mu}\frac{dp}{dx} \tag{6}$$

Eliminating Q between (6) and (2), we find

$$\frac{dp}{dx} = -\frac{16\mu\kappa x}{R^3} \tag{7}$$

so that

$$dp = -\frac{16\mu\kappa\, xdx}{R^3} \tag{8}$$

Integrating (8) from the center-point (x=0) to atmospheric (x=L), we get

$$p(0) - p(L) = \frac{8\mu\kappa L^2}{R^3} \tag{9}$$

C. There are three criteria,

$$\frac{R}{L} << 1, \quad \frac{\rho U R}{\mu}\frac{R}{L} << 1, \quad \frac{R^2}{\nu \Delta t} << 1 \tag{10}$$

in which U should be chosen conservatively as the highest velocity in the system, $U \approx 2\kappa L/R$, yielding, to order of magnitude,

$$\frac{R}{L} << 1, \quad \frac{\kappa R}{\nu} << 1, \quad \frac{R^2}{\nu \Delta t} << 1 \tag{11}$$