2.25 QUIZ 2, 13 NOVEMBER 2003

ANSWER TO PROBLEM 1

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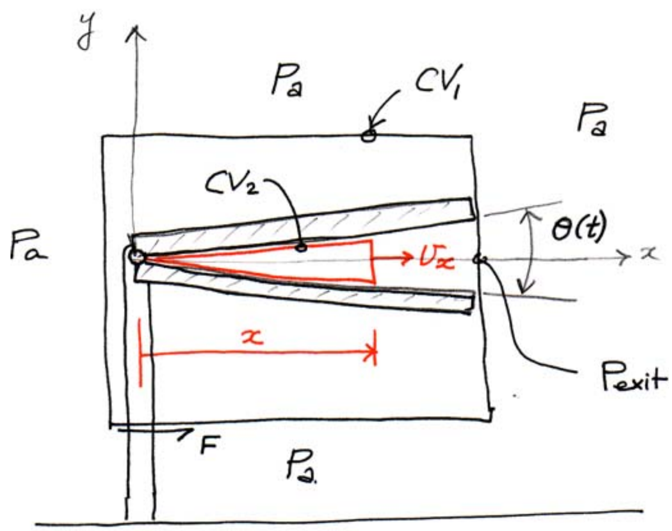
<u>NOTE</u>: All extensive quantities are *per unit width*, consistent with a 2D problem.
The answer given here for part (b) is much more complete than was required for the quiz.



Momentum theorem applied to the fixed CV_1:

$$\frac{d}{dt}\int_{CV_1} \rho v_x v_{rn} dV + \int_{CS_1} \rho v_x v_{rn} dA = (F_x)_{CV_1} \tag{1}$$

First, to obtain the outflow velocity $v_x(x,t)$ between the plates, we apply mass conservation to CV_1:

$$\frac{d}{dt}\int_{CV_2} \rho dV + \int_{CS_2} \rho v_{rn} dA = 0 \tag{2}$$

During the **closing stroke stroke**,

$$\frac{d\theta}{dt} = -\omega \tag{3}$$

and eq. (2) takes the form

$$-\frac{d}{dt}\left(\frac{x\cdot x\theta}{2}\right)+v_x\cdot x\theta=0 \tag{4}$$

from which, using (3),

$$v_x(x,t)=\frac{\omega x}{2\theta(t)} \tag{5}$$

Substituting next from (5) into (1), we have

$$\frac{d}{dt}\int_0^L \rho\left(\frac{x\omega}{2\theta}\right)\cdot x\theta\cdot wdx+\rho\left(\frac{L\omega}{2\theta}\right)^2\cdot\theta Lw=F+(p_a-p_{exit})\theta L \tag{6}$$

F is the thrust force exerted in the x-direction on the CV at the "cut" through the supports, which is equal to the thrust force in the –x direction. The second term on the right accounts for the possibility that the pressure at the exit plane differs from the atmospheric (see below). During the closing stroke this term is zero, there being outflow from the gap. The first term on the left in (6) is also zero, since the two $\theta(t)$s cancel out and leave the integrand without time dependence. For the closing stroke the momentum theorem thus gives

$$F_{close}=\rho\left(\frac{L\omega}{2\theta}\right)^2\cdot\theta Lw=\frac{\rho\omega^2L^3w}{4\theta(t)} \tag{7}$$

where

$$\theta=\theta_1-\omega t \tag{8}$$

For the **opening stroke**, the flow is in the reverse direction,

$$v_x(x,t)=-\frac{\omega x}{2\theta(t)}, \tag{9}$$

and there is a Bernoulli pressure drop $\frac{1}{2}\rho V^2$ from outside the gap to the gap's entrance/exit plane. This changes F to

$$F_{open}=\rho\left(\frac{L\omega}{2\theta}\right)^2\cdot\theta Lw-\frac{\rho V^2}{2}\theta Lw=\frac{\rho\omega^2L^3w}{8\theta} \tag{10}$$

where

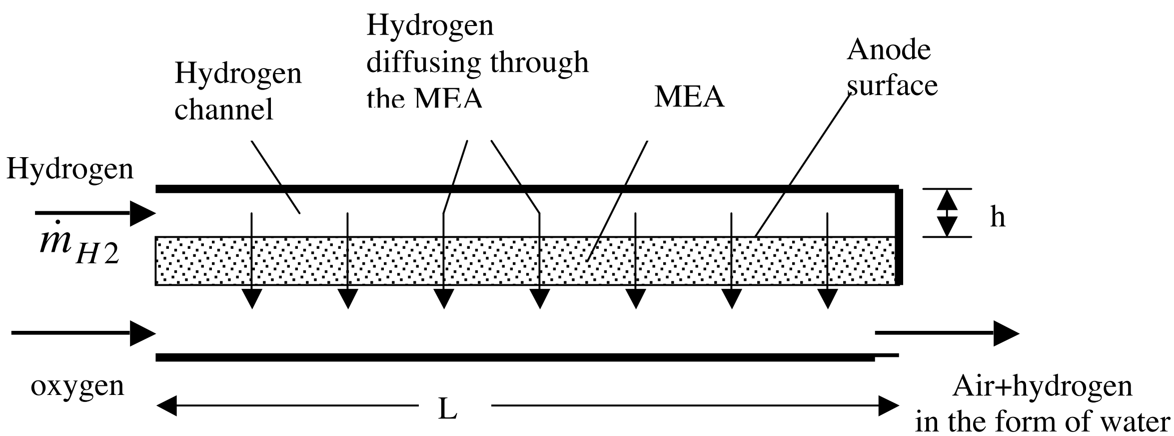
$$V=\frac{\omega L}{2\theta(t)} \tag{11}$$

is the velocity at the inlet to the gap and

$$\theta = \theta_2 + \omega t. \tag{12}$$

The opening stroke produces a *forward* thrust, like the closing stroke (negative momentum flowing into the CV has the same sign as positive momentum flowing out of it), but due to the suction resulting from the Bernoulli effect, the thrust during opening is a factor of two smaller than the thrust during the closing stroke at the same angle. Clearly, the system produces thrust. The propulsion efficiency can be determined based on the input and output powers.

PROBLEM II



(i) $h/L = 10^{-2}$ and $Uh^2/(\upsilon L) \sim 0.025$, the flow can be treated as locally fully developed.

(ii) The flow through the hydrogen channel is locally fully developed, and hence the volume flow rate per unit depth is given by:

$$Q' = -\frac{h^3}{12\mu}\frac{dp}{dx}.$$

The total mass flow rate locally is given by: $\dot{m} = -\dfrac{wh^3}{12\mu}\rho\dfrac{dp}{dx} = -\dfrac{w\alpha h^3}{24\mu}\dfrac{dp^2}{dx}$,

where we used the equation of state $\rho = \alpha p$.

Now, we apply mass conservation over a differential control volume at an arbitrary point in the channel:

$$\frac{d\dot{m}}{dx} + w\dot{m}_a = 0,$$

and use the expression above for the mass flow rate to get:

$$\frac{d^2 y}{dx^2} - a^2 x = 0$$

where $y = p^2$ and $a^2 = \frac{24\mu\beta}{\alpha h^3}$. And we used $\dot{m}_a = \beta p^2$.

The boundary conditions are: at $x = 0, p = p_0$ and at $x \to L(\to \infty), p \to 0.$ Note that the second simplification is used since $L/h >> 1$.

The solution can then be written as:

$$p^2 = p_0^2\, e^{-ax}.$$

(iii) The total mass flow is $\dot{m}_{H_2} = w\int_0^L \beta p^2 dx = \frac{w\beta p_0^2}{a}\left(1 - e^{-aL}\right) \approx \frac{w\beta p_0^2}{a}$,

or $p_0 = \sqrt{\frac{24\mu\,\dot{m}_{H_2}}{\alpha\, h^3}}$.

Note that the pumping power $\dot{m}_{H_2} p_0 = \sqrt{\frac{24\mu}{\alpha h^3}}\dot{m}_{H_2}^{3/2}$, while the power output is proportional to $\dot{m}_{H_2}$.