## 2.25 ADVANCED FLUID MECHANICS Fall 2004

FINAL EXAM SOLUTIONS
Each problem is graded out of 20 points
(40 points total)

Please see Prof Sonin with grading queries regarding Question 1
Please see Prof McKinley with grading queries regarding Question 2

# SOLUTION: PROBLEM 1 (Ain Sonin)

PART A

From the definitions of the lift and drag coefficients in the problem statement,

$$Lift = 2\int_0^R L'dr = 2\int_0^R c_L \tfrac{1}{2}\rho(\omega r)^2 bdr = \frac{c_L\rho\omega^2 bR^3}{3} \qquad (1)$$

PART B

Equating lift to the total helicopter weight *Mg* that is kept aloft, we obtain, using Part A, that the required rotation rate is

$$\omega_* = \sqrt{\frac{3Mg}{c_L\rho bR^3}} \qquad (2)$$

PART C

Apply the steady-state angular momentum theorem,

$$\int_{CS} \rho r V_\theta V_{rn} dA = (T_z)_{CV} , \qquad (3)$$

to a control volume that envelops the rotor blades and jet engines and "cuts" through the frictionless bearing and the exit planes of the two jet engines. At the jet exit planes, the $\theta$-direction gas velocity *relative to the inertial reference frame of the ground* is $V_\theta = \omega R - V_0$, and $V_{rn} = V_0$. The torque due to aerodynamic drag may be deduced from the formula for drag given in the problem statement:

$$(T_z)_{CV} = -2\int_0^R c_D \tfrac{1}{2}\rho(\omega r)^2 brdr = -\frac{c_D\rho\omega^2 bR^4}{4} \ . \qquad (4)$$

Inserting this into (3), and using (2) for the angular rotation rate, we cast (3) into the simple form

$$V_0 = \omega R + \frac{3}{4}\frac{c_D}{c_L}\frac{Mg}{\rho_0 V_0 A_0} \qquad (5)$$

where $A_0$ is the exit area of a jet. Equation (5) reduces to a quadratic equation for $V_0$, which can be solved straightforwardly. Putting the equation in the form (5) is useful since it shows at a glance that the torque due to aerodynamic drag becomes negligible in the limit

$$\frac{3}{4}\frac{c_D}{c_L}\frac{Mg}{\rho_0 V_0 A_0} << \omega R \qquad (6)$$

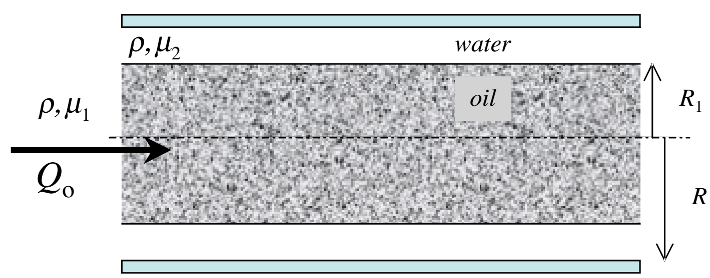
PART D

The power required for hovering flight is the torque exerted on the two rotor blades —Eq. (4)—multiplied by $\omega$.

**NOTE:**

The above solution accounts for the dynamic effects of the jet exit velocity but neglects the effects of the inflow into the jet engines. This is a reasonable approximation: the hot exhaust gases are much less dense than the cool inlet air. Mass conservation implies that, with inlet and exit areas of the same order of magnitude, both the velocity and the momentum flux will be much higher at the exit than in the cool gas at the inlet.

## Problem 2: Lubricated Pipelining (Gareth McKinley)




Model the fluid motion as flow in a cylindrical pipe of radius $R$ with a *core* of thickness $R_1$ consisting of viscous liquid oil with viscosity $\mu_1$ surrounded by a shell of water or other low viscosity fluid) of thickness $\delta = R - R_1$ that is density matched (so that $\rho_1 = \rho_2 = \rho$) with viscosity $\mu_2 < \mu_1$. The interfacial tension between the two liquids is denoted $\sigma$. The average velocity of the oil through the pipe is denoted $\overline{v}_o = Q_o / \pi R_1^2$

a) Dimensional Analysis:

$$\Delta P / L = f(\rho, \overline{v}_o, R_1, \mu_1, \mu_2, R, \sigma) \qquad (1)$$

Hence $n = 8$; $r = 3$; $(n - r) = 5$ Pi groups. Note that only (any) two of the length scales can enter as they are constrained by $R_1 = R - \delta$. Gravity does not enter since there is no density contrast to drive density waves at interface ($\Delta\rho = \rho_1 - \rho_2 = 0$). Pick as primary variables the average oil velocity $\bar{v}_o = Q_o / \pi R_1^2$ (flow), core radius $R_1$ (geometry) and density $\rho$. Obtain following Pi groups:

$$\frac{\Delta P / L}{\left(\rho \bar{v}_o^2 / R_1\right)} = \phi\left(\frac{\mu_1}{\rho \bar{v}_o R_1}, \frac{\mu_2}{\rho \bar{v}_o R_1}, \frac{R}{R_1}, \frac{\sigma}{\rho \bar{v}_o^2 R_1}\right) \tag{2}$$

where the dimensionless groups may be recognized as, respectively, a friction factor, Reynolds numbers for the inner and outer fluids, a geometric ratio and the *Weber number* (inertia/capillarity) $We = \rho \bar{v}_o^2 R_1 / \sigma$. Alternatively you may have picked the inner fluid viscosity as the third primary variable (characterizing the flow), in which case you obtain:

$$\frac{\Delta P / L}{\mu_1 \bar{v}_o / R_1^2} = \phi\left(\frac{\rho \bar{v}_o R_1}{\mu_1}, \frac{\mu_2}{\mu_1}, \frac{R}{R_1}, \frac{\sigma}{\mu_1 \bar{v}_o}\right) \tag{3}$$

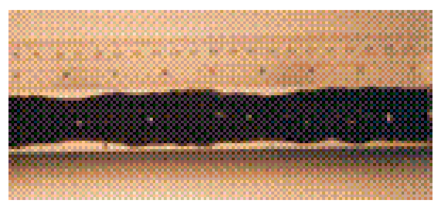
Where the last group is inverse of capillary number. Note that $Ca = We/Re$. In general, the way this problem would be written "by inspection" would be in terms of a friction factor as a function of Reynolds number and other characteristic ratios:

$$f = \phi\left(Re_1, \mu_2/\mu_1, R_1/R, We\right) \tag{4}$$

Under flowing conditions, surface tension is the stabilizing force which resists the formation of curved interfaces (i.e. this would lead to extra interfacial area which is energetically unfavorable). Hence we require

$$We = \rho \bar{v}_o^2 R_1 / \sigma << 1 \tag{5}$$

This is actually not easy to achieve (for water/oil with $\sigma \approx 30 \times 10^{-3} N/m$ and $R_1 \approx 1m$ we would require $\bar{v}_o \leq$ 5.5mm/s). So typical lubricated pipelining operations do tolerate interfacial waves – provided they do not "break" (i.e. form crests that overturn and would thus emulsify the oil).



http://www.aem.umn.edu/research/pipeline/horizontalindex.html

b) The appropriate boundary condition for the shear stress on the interface $r = R_1$ assuming the interface is cylindrical can be written.

$$\tau_{rz}\big|_{r=R_1^-} = \tau_{rz}\big|_{r=R_1^+} \quad \text{or} \quad \mu_1 \frac{\partial v_z}{\partial r}\bigg|_{r=R_1^-} = \mu_2 \frac{\partial v_z}{\partial r}\bigg|_{r=R_1^+} \tag{6}$$

The capillary pressure increase across the interface gives $p_1(z) = p_2(z) + \dfrac{\sigma}{R_1}$. This increase is negligible if the capillary pressure correction is small compared to the hydrostatic change from top to bottom that we are also neglecting; *i.e.* $\sigma/R_1 << \rho g(2R_1)$ or $\sigma/(2\rho g R_1^2) << 1$. Even if this term is included, it does not change the conclusion that $\dfrac{\partial p_2}{\partial z} = \dfrac{\partial p_1}{\partial z} = -\dfrac{\Delta P}{L}$.

c) Assuming that the steady-state velocity field is $\boldsymbol{v} = \left[0, 0, v_z(r)\boldsymbol{\delta}_z\right]^T$ and substituting into the $z$-component of Navier-Stokes equations gives

$$0 = -\frac{\partial p}{\partial z} + \frac{\mu_1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) \quad 0 \le r \le R_1$$
$$0 = -\frac{\partial p}{\partial z} + \frac{\mu_2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) \quad R_1 \le r \le R \tag{7a,b}$$

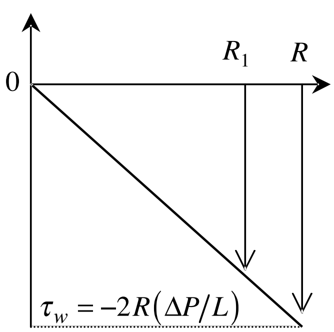
Using these boundary conditions together with the condition that there is a finite interfacial velocity given by $v_i = v_z(r = R_1^-) = v_z(r = R_1^+)$ gives the following expressions for the fully-developed velocity field $v_z(r)$:

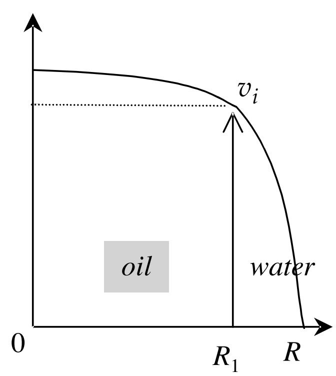
$$v_z^{(1)} = \frac{1}{4\mu_1}\left(\frac{\Delta P}{L}\right)\left[R_1^2 - r^2\right] + \frac{1}{4\mu_2}\left(\frac{\Delta P}{L}\right)\left[R^2 - R_1^2\right] = \frac{1}{4\mu_1}\left(\frac{\Delta P}{L}\right)\left[R_1^2 - r^2\right] + v_i \quad \text{for } 0 \le r \le R_1 \tag{8a}$$

$$v_z^{(2)} = \frac{1}{4\mu_2}\left(\frac{\Delta P}{L}\right)\left[R^2 - r^2\right] \quad \text{in the shell from } R_1 \le r \le \text{R}. \tag{8b}$$

The value of the velocity at the interface is given by:

$$v_z^{(2)}(r = R_1) = v_z^{(1)}(r = R_1) \equiv v_i = \frac{1}{4\mu_2}\left(\frac{\Delta P}{L}\right)\left[R^2 - R_1^2\right] \tag{9}$$



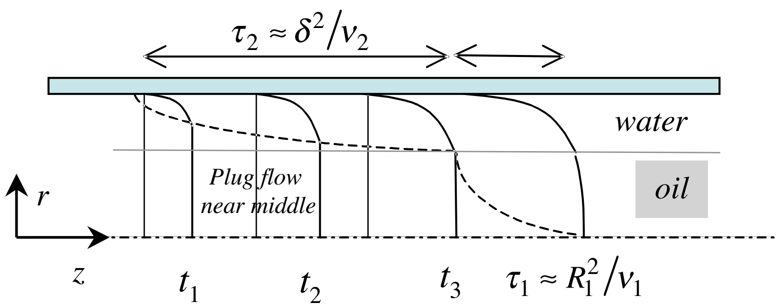




The shear stress is <u>negative</u> and increases linearly in magnitude across the entire pipe (*independent* of viscosity or if flow is laminar or turbulent!);

$$\tau_{rz}(r) = \frac{r}{2}\frac{\partial p}{\partial z} = -\frac{r}{2}\left(\frac{\Delta P}{L}\right)$$

The velocity field is continuous but changes slope by a factor of $(\mu_1/\mu_2)$ at the core/shell boundary $r = R_1$ .

d) The flow in the pipeline is typically started impulsively by imposing a sudden increase in the pressure gradient along the pipe, and the flow takes a period of time to become fully developed. The center of the pipe moves as a 'plug' until viscous effects diffuse in from the no-slip boundary.



Engineering estimates of the total time taken for the flow field in each domain to reach steady state are thus:

$$\tau_2 \approx \frac{\delta^2}{\nu_2} = \frac{\rho(R-R_1)^2}{\mu_2} \qquad\qquad \tau_1 \approx \frac{R_1{}^2}{\nu_1} = \frac{\rho R_1{}^2}{\mu_1} \tag{10}$$

typically $\mu_2 << \mu_1$ whereas $\delta, R_1$ might be only a factor of two or three different so the first inequality dominates. The cross-over for these inequalities comes when $\delta^2/R_1{}^2 > \mu_2/\mu_1$ .

e) The volume flow rate of each component is given by:

$$\begin{aligned} Q_o = \int_0^{R_1} 2\pi r v_z^{(1)} dr &= \frac{\pi}{2}\left(\frac{\Delta P}{L}\right)\int_0^{R_1} \frac{r}{\mu_1}\left[R_1{}^2 - r^2\right] + \frac{r}{\mu_2}\left[R^2 - R_1{}^2\right] dr \\ &= \frac{\pi}{4\mu_2}\left(\frac{\Delta P}{L}\right) R_1{}^2\left[R^2 - R_1{}^2\right] + \frac{\pi}{8\mu_1}\left(\frac{\Delta P}{L}\right) R_1{}^4 \\ &= \pi R_1{}^2 v_i \; + \; \frac{\pi}{8\mu_1}\left(\frac{\Delta P}{L}\right) R_1{}^4 \end{aligned} \tag{11}$$

The last line in eq.(11) shows that the oil flow rate consists of the usual Poiseuille result (second term) PLUS an additional contribution from the core with a finite interfacial velocity.
The water flow rate is:

$$Q_w = \int_{R_1}^{R} 2\pi r v_z^{(2)} dr = \frac{\pi}{8\mu_2}\left(\frac{\Delta P}{L}\right)\left[R^2 - R_1{}^2\right]^2 \tag{12}$$

Using these results, the total power dissipated by the pumping operation per unit length is:

$$\dot{W}_L = \left(\frac{\Delta P}{L}\right)(Q_o + Q_w) = \frac{\pi}{8}\left(\frac{\Delta P}{L}\right)^2 \left\{\frac{1}{\mu_2}\left(R^2 - R_1{}^2\right)^2 + \frac{2}{\mu_2} R_1{}^2\left(R^2 - R_1{}^2\right) + \frac{1}{\mu_1} R_1{}^4\right\}$$

f) Differentiating the expression for $Q_o$ with respect to the core radius $R_1$ gives:

$$0 = \frac{\partial Q_o}{\partial R_1} = \frac{\pi}{4\mu_2}\left(\frac{\Delta P}{L}\right)\left[2R^2 R_1 - 4R_1{}^3\right] + \frac{\pi}{8\mu_1}\left(\frac{\Delta P}{L}\right)\left(4R_1{}^3\right) \tag{13}$$

assuming the viscosities $\mu_1, \mu_2$ and density $\rho$ are all held constant and that there is a fixed value of the imposed pressure gradient $\Delta P/L$. Solving gives:

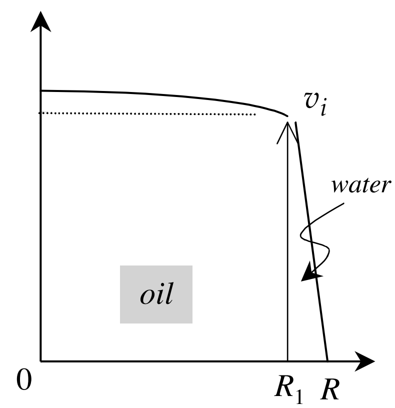
$$R_1{}^* = \frac{R}{\left[2 - \mu_2/\mu_1\right]^{1/2}} \tag{14}$$

for the optimal value of the core radius that maximizes the volume flow rate of oil through the pipe. In the limit $\mu_2/\mu_1 << 1$ this value approaches $R_1^* \approx R/\sqrt{2}$ . A smaller core radius (2nd term in eq. 11) cuts down on the amount of oil transported; a larger core means a smaller outer shell and an increase in the dissipation within the water phase.

Optional Extra Credit: (more detail provided than needed)

g) In the limit that the outer layer of fluid becomes very thin and of very low (but still non-zero!) viscosity ($\mu_2 << \mu_1$) then we can approximate $R_1{}^2 = (R-\delta)^2 \approx R^2 - 2R\delta$ and equation (9) for the interface velocity becomes

$$v_i = \frac{\delta}{\mu_2}\left\{\frac{R}{2}\left(\frac{\Delta P}{L}\right)\right\} = \frac{\delta}{\mu_2}\tau_w \qquad (15)$$




where the term in { } can be recognized from the momentum equation (or from a simple control volume force balance of the form $\tau_w(2\pi RL) = \Delta P(\pi R^2)$) as the wall shear stress $\tau_w$. The type of equation $v_{slip} = \beta\tau_w$ in eq.(15) is known as the *Navier slip law* . The slip coefficient is thus $\beta = \delta/\mu_2$ . The magnitude of the slip coefficient thus depends on the rate at which these two parameters go to zero.

Alternately: note that when the gap is very small then the flow in the outer (water) phase becomes almost a Couette flow. To see this linearize equation 8b using a new variable that starts at the wall ($y = R - r$) to get $r^2 \approx R^2 - 2Ry$ and thus

$$v_z^{(2)}(y) \approx \frac{1}{4\mu_2}\left(\frac{\Delta P}{L}\right)2Ry$$

which again looks like a Couette flow introduced in class with $\tau = \mu_2 \frac{\partial v_z}{\partial r} = -\mu_2 \frac{\partial v_z}{\partial y} = \frac{R}{2}\left(\frac{\Delta P}{L}\right)$ and a maximum velocity at the edge of the Couette 'shell' layer (i.e. at position $y = \delta$) given by

$$v_z^{(2)}(y=\delta) \approx \frac{R}{2\mu_2}\left(\frac{\Delta P}{L}\right)\delta = \frac{\delta}{\mu_2}\tau_w$$