2.25 ADVANCED FLUID MECHANICS Fall 2005

Final Exam

FRIDAY, December 16, 2005, 1:30 – 4:30 P.M.

OPEN EXAM WHEN TOLD AT 1:30 PM

THERE ARE TWO PROBLEMS
OF EQUAL WEIGHT

Please answer each question in SEPARATE books

This exam book consists of SIX (6) pages

You may use ELEVEN (11) pages of handwritten notes as well as the single page handouts posted on MIT server plus any tables you feel are necessary

2. Flow Focusing in Microfluidics [40 points]

There is a lot of interest at present in using microfluidic devices to produce very fine scale droplets which are also all monodisperse (i.e. the same size). To do this, two immiscible fluids (*i.e.* the fluids don't mix) are brought together via suitable inlet tubing and then special geometries are designed which focus the fluid motion down to very small scales. Typically two fluids are used with different viscosities. In what follows we will always label the inner fluid *'i'* and the outer fluid *'o'*. The inner fluid is usually more viscous than the outer one ($\mu_i \geq \mu_o$).

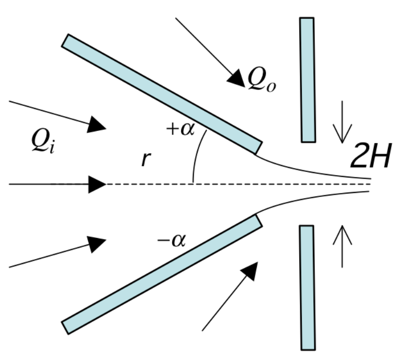
Two representative geometries are shown in the figures below. On the left is a planar flow-focusing geometry consisting of a converging channel (with opening angle α) in which fluid is pushed under pressure from left to right. On the right is an axisymmetric flow-focusing geometry in which the two fluids are sucked (via a negative gage pressure) into the cylindrical tube. In each case a long thin thread of the inner fluid is formed, which ultimately breaks up into monodisperse droplets. In this question we first consider some generic features of these problems from the point of view of dimensional analysis and then analyze these two geometries sequentially.

Case I: Flow focusing in two dimensions

Case II: Axisymmetric flow focusing

A. Dimensional Analysis of Flow Focusing [10 pts]

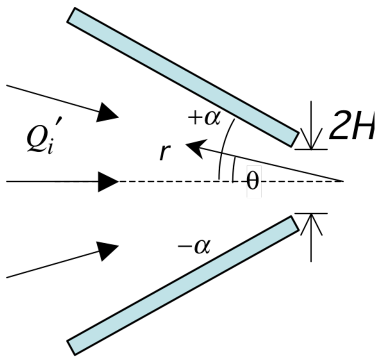
(i) Let us consider a generic situation as shown in the sketch below. Two fluids (with identical densities $\rho_i = \rho_o = \rho$) are brought together with volumetric flow rates Q_i and Q_o with interfacial tension σ in a geometry with characteristic gap or orifice size $2H$ and an entrance angle α. We seek to understand how the thread radius R varies with the parameters in the problem. Use dimensional analysis to deduce what dimensionless groups control the thread radius



Because the flows are on the small scale, we expect viscous effects to be important; so you should use μ_i as one of the primary variables in your analysis.

(ii) The breakup of the final inner fluid thread is very complicated because inertia, viscosity and surface tension may all play a role. Use dimensional analysis to show that there is a unique characteristic length (denoted ℓ_c) for the inner fluid thread that depends only its material properties. Evaluate this characteristic length scale for water (you should and for glycerol ($\mu_i = 1\text{Pa.s}$, $\rho_i = 1000\text{kg/m}^3$, $\sigma = 0.062\text{N/m}$).

B. Flow in a Planar Converging Channel [15pts]



We now consider a simplified geometry to model the first process shown above; i.e. the converging flow of the *inner fluid* confined between two solid walls inclined at angles $\pm\alpha$ to the horizontal. We will model this flow as a planar two dimensional flow. Use a cylindrical polar coordinate system $\{r,\theta\}$ as shown in the sketch. For distances $r \geq H/\sin\alpha$ the flow is purely radial.

(i) First show using an appropriate conservation equation that the velocity field can be satisfied by a functional form $v_r = -f(\theta)/r$. What are the units of the function *f*?

Give an expression for the volumetric flowrate (per unit depth) Q_i' through the converging 2D nozzle at any position *r*.

(ii) Write down and simplify the radial and tangential components of the Navier-Stokes equations to find expressions for the radial and tangential pressure gradients.

(iii) Use the equality of cross derivatives (i.e. $\frac{\partial^2 p}{\partial r \partial \theta} = \frac{\partial^2 p}{\partial \theta\, \partial r}$) to eliminate the unknown pressure field. Non dimensionalize your equation using Q_i' to scale the function f to show that the velocity field is governed by the solution of an equation with the form

$$\frac{d^3F}{d\theta^3} - \Re F\frac{dF}{d\theta} + 4\frac{dF}{d\theta} = 0$$

where F is the dimensionless form of *f*. Also specify the boundary conditions needed to solve this equation. What is the dimensionless group $\Re$ and what does it measure?

(iv) Show that in the limit $\Re \to 0$ this equation can be solved by a function of the form $F(\theta) = A + B\sin 2\theta + C\cos 2\theta$ where *A, B, C* are constants. Determine the constants and find a final expression for the velocity field in the converging channel.

(v) Does the pressure in the fluid increase or decrease as one approaches the narrow apex of the channel?

C) Flow Focusing into a Capillary Tube [15pts]

Now let us focus on the axisymmetric case (i.e. Case II in figure 1) with a spherical polar coordinate system $\{r,\theta,\phi\}$ located with an origin at the middle of the inlet tube.

(i) If viscous effects are negligible then we can model the fluid motion as a steady inviscid flow. By evaluating and sketching the velocity field for a point doublet with velocity potential given by

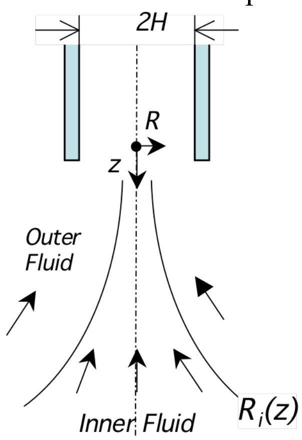
$$\phi = -\frac{m}{r^2}\cos\theta \qquad \text{(C1)}$$

show that this is a reasonable description of the fluid flow pattern shown in the picture for $r \geq H$. Give an equation for the streamlines (or strictly *streamtubes* as this is axisymmetric).

ii) Evaluate the pressure field in the converging flow approaching the entrance to the tube.

iii) In reality, viscous effects in the inner fluid cannot be ignored. Write down the appropriate tangential and normal boundary conditions to use at the interface between the two fluids.

iv) It is clear from the picture that the flow focusing leads to a *slender* converging thread of the inner fluid. If we use a cylindrical coordinate system as shown in the sketch opposite then the equation for the interface is $R_i = Kz^{3/2}$ where K is a constant. Use a lubrication analysis to find the velocity field in the inner fluid, assuming the outer fluid has negligible viscosity ($\mu_o = 0$). Clearly state any assumptions that you make in your solution



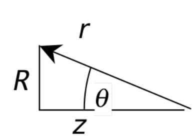
(Generous hint: if you are confused about your solution, justify the form of this velocity field by evaluating the vorticity of the inner fluid).

v) In reality, the outer fluid is **not** inviscid and a boundary layer develops in the outer fluid. Write down the appropriate boundary layer equations for the velocity field in the outer fluid and give all of the appropriate boundary conditions. You do not have to solve these!! Do you expect the boundary layer to grow more rapidly or more slowly than the Blasius solution for a flat plate; Give TWO reasons to support your conclusion.

Some useful identities

From reading Kundu Chapter 6 you will remember that for axisymmetric problems, both spherical $\{r,\theta,\phi\}$ and cylindrical $\{R,\phi,z\}$ polar coordinates are useful; with $z = r\cos\theta$ and $R = r\sin\theta$. The following identities may be helpful to you:

In spherical polar coordinates:



gradient of a scalar *S*; $\quad \nabla S = \frac{\partial S}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial S}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial S}{\partial \phi}\mathbf{e}_\phi$

Stream tubes $\psi(r,\theta)$ are given by $\quad v_r = \frac{1}{r^2\sin\theta}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \theta}$

The azimuthal vorticity in cylindrical coordinates is $\omega_\phi = \frac{\partial v_R}{\partial z} - \frac{\partial v_z}{\partial R}$