System Identification

6.435

SET 4

- –Input Design
- –Persistence of Excitation
- –Pseudo-random Sequences

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Input Signals

- Commonly used signals
  - Step function
  - Pseudorandom binary sequence (PRBS)
  - Autoregressive, moving average process
  - Periodic signals: sum of sinusoids
- Notion of "sufficient excitation". Conditions !
- Degeneracy of input design.
- Relations between PBRS & white noise.
- Frequency domain properties of such signals.

Examples

- Step input

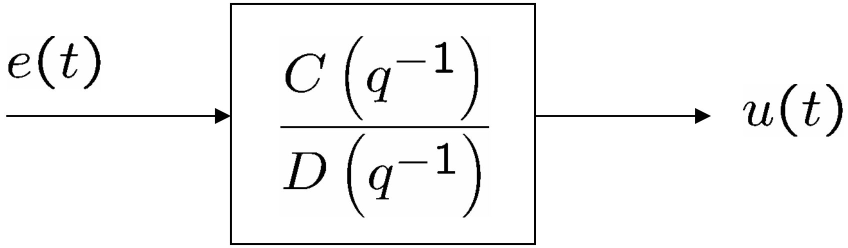
$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{other} \end{cases}$$

- A Pseudorandom binary sequence
  - periodic signal
  - switches between two levels in a certain fashion
  - levels = $\pm a$ period = M

- Autoregressive moving average

$e(t)$ is a random sequence

$$\frac{1}{N}\sum e(t)e(t-\tau) \rightarrow 0 \quad N \rightarrow \infty \quad \tau \neq 0$$



{ARMA process}

- $u(t) = \sum_{u=1}^{m} a_j \sin(\omega_j t + \phi_j)$

$$0 \leq \omega_1 < \omega_2 \quad \ldots \quad < \omega_m \leq \pi$$

Spectral Properties

- PRBS

$$x(k+1) = \begin{pmatrix} a_1 & \dots & a_n \\ 1 & & \\ & \ddots & \\ 0 & \dots & 10 \end{pmatrix} x(k)$$

$$y(k) = (0 \dots 1) x(k)$$ [takes on 0,1]

all calculations are mod 2.

$a_1, \dots, a_n$ are either 0 or 1

• Covariance function

$$R_u(\tau) = \begin{cases} a^2 & \tau = 0, \pm M, \pm 2M, \ldots \\ -\frac{a^2}{M} & \tau = \text{other} \end{cases}$$

$$\Phi_u(\omega) = \sum_{\tau=-\infty}^{\infty} R_u(\tau) e^{-i\omega\tau}$$

$$= \sum_{k=0}^{M-1} C_k \delta\left(\omega - \frac{2\pi k}{M}\right)$$

and evaluating C_k,

$$\Phi_u(\omega) = 2\pi \frac{a^2}{M^2} \left[\delta(\omega) + (M+1) \sum_{k=1}^{M-1} \delta\left(\omega - \frac{2\pi k}{M}\right) \right]$$

- ARMA

$$\Phi_u(\omega) = \lambda^2 \frac{\left|C\left(e^{i\omega}\right)\right|^2}{\left|D\left(e^{i\omega}\right)\right|^2}$$

- Sum of sinusoids

$$R_u(\tau) = \sum_{j=1}^{m} \frac{{a_j}^2}{2} \cos(\omega_j \tau)$$

$$\Phi_u(\omega) = \sum_{j=1}^{m} \frac{{a_j}^2}{4} [\delta(\omega - \omega_j) + \delta(\omega + \omega_j)]$$

PRBS vs. WN

- Given any smooth function

$$I_1 = \int_{-\pi}^{\pi} f(\omega)\Phi_u(\omega) = \left(\frac{a^2}{M^2} f(0) + \frac{a^2(M+1)}{M^2} \sum_{k=1}^{M-1} f\left(\frac{2\pi}{M}k\right)\right) \cdot 2\pi$$

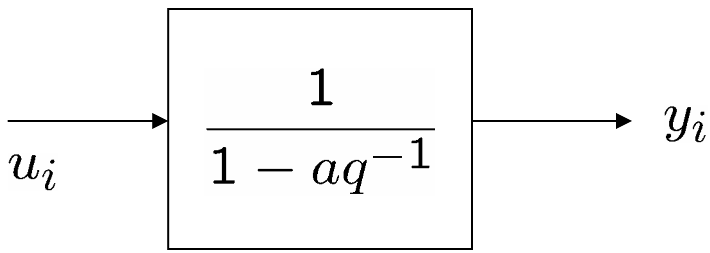
$$I_2 = \int_{-\pi}^{\pi} f(\omega)\lambda^2 d\omega = \lambda^2 \int_{-\pi}^{\pi} f(\omega) d\omega$$

- Approximate the integral $\int_{-\pi}^{\pi} f(\omega) d\omega$ by Riemann sum.

$$\Rightarrow I_1 \simeq I_2$$

- The spectrum of PRBS approximate WN as distributions.

- <u>Homework</u>:



u_1 = WN
u_2 = PRBS

Compare R_{y_1}, R_{y_2}.

Persistent Excitation

Definition:

A quasi-stationary input, u, is persistently exciting of order n if the matrix

$$\bar{R}_n = \begin{bmatrix} R_u(0) & \dots & R_u(n-1) \\ & & \\ R_u(n-1) & \dots & R_u(0) \end{bmatrix}$$

is positive definite.

- Recall: The correlation method in the time-domain required the inversion of $\bar{R}_M$ to estimate M-parameters of the impulse response.

Relation to the Spectrum

Theorem:

Let u be a quasi-stationary input of dimension nu, with spectrum $\Phi_u(\omega)$. Assume that $\Phi_u(\omega) > 0$ for at least n distinct frequencies. Then u is p.e. of order n.

Proof:

Let $g^T = (\underbrace{g_1}_{nu}, \ldots, \underbrace{g_n}_{nu})$ be a n $x nu$ row vector such

that $g^T R_n g = 0$. Define $G\left(q^{-1}\right) = \sum_{i=1}^{n} g_i q^{-i}$.

Then,

$$0 = g^T R_n g = E\left[\left(G\left(q^{-1}\right) u\right)\left(G\left(q^{-1}\right) u\right)^T\right]$$

$$= \int_{-\pi}^{\pi} \underbrace{G\left(e^{i\omega}\right) \Phi_u(\omega) G^T\left(e^{-i\omega}\right)}_{\geq 0} dw$$

$\Rightarrow G\left(e^{i\omega}\right) \Phi_u(\omega) G^T\left(e^{-i\omega}\right) = 0$. But $\Phi_u(\omega) > 0$ at n distinct frequencies $\Rightarrow G(e^{i\omega}) = 0$ at these frequencies = g = 0.

Theorem (Scalar):

If u is p.e of order $n \Rightarrow \Phi_u\left(e^{i\omega}\right) \neq 0$ for at least n-points.

Proof:

Suppose $\Phi_u(\omega) \neq 0$ for at most $(n-1)$ points.

Let g be any vector with $G\left(q^{-1}\right) = \sum_{i=1}^{n} g_i q^{-i}$.

$g^T R_n g = 0 \Leftrightarrow \left|G\left(e^{i\omega}\right)\right| \Phi(\omega) = 0$

Pick a vector $g \quad \exists$

$$\left|G\left(e^{i\omega}\right)\right| = 0 \quad \text{at the } (n-1) \text{ points where } \Phi(\omega) \neq 0$$

$$\left|G\left(e^{i\omega *}\right)\right| \neq 0 \quad \text{at some other frequency.}$$

then $g \neq 0 \quad \& \quad g^T R_n g = 0 \quad \Rightarrow\Leftarrow$

Examples

- Step input: persistently exciting of order 1
- PRBS: $n \leq M$.

$$R_u(\tau) = \begin{cases} a^2 & \tau = 0, \pm M, \pm 2M, \ldots \\ -\frac{a^2}{M} & \text{otherwise} \end{cases}$$

$$R_n = \begin{pmatrix} a^2 & \frac{-a^2}{M} & \ldots & \frac{-a^2}{M} \\ \frac{-a^2}{M} & a^2 & \ldots & \frac{-a^2}{M} \\ & & \ddots & \\ \frac{-a^2}{M} & & \ldots & -a^2 \end{pmatrix}$$

det $R_n \neq 0$.
(Verify!)

$$R_{M+1} = \begin{pmatrix} a^2 & \frac{-a^2}{M} & \cdots & \frac{-a^2}{M} & a^2 \\ \frac{-a^2}{M} & a^2 & \cdots & & \frac{-a^2}{M} \\ & & \ddots & & \\ a^2 & \frac{-a^2}{M} & \cdots & \frac{-a^2}{M} & a^2 \end{pmatrix} \longrightarrow$$

singular
1st and last row are the same.

PRBS is p.e. of order M.

- ARMA Process is p.e. of any order.

- Sum of sinusoids

$$u(t) = \sum_{j=1}^{m} a_j \cos(\omega_j t + \phi_j)$$

$$0 \leq \omega_1 < \omega_2 \quad \ldots \quad < \omega_n \leq \pi$$

$$\Phi_u = \sum_{j=1}^{m} \frac{a_j}{2} [\delta(\omega - \omega_j) + \delta(\omega + \omega_j)]$$

Φ_u is non-zero at exactly n-points, where

$$n = \begin{cases} 2m & 0 < \omega_1, \omega_n < \pi \\ 2m-1 & 0 < \omega_1 \text{ xor } \omega_n = \pi \\ 2m-2 & 0 = \omega_1 \ \& \ \omega_n = \pi \end{cases}$$

Theorem:

$u(t)$ is quasi-stationary. Define $z(t) = \sum_{i=1}^{n} H_i u(t-i)$

Then $\bar{E} z(t) z(t)^T = 0 \Rightarrow H_i = 0 \quad i = 1, \ldots, n$ iff

$u(t)$ is persistently exciting of order $\boldsymbol{n}$.

Proof:

Define

$$\bar{H} = (H_1, \ldots, H_n)^T$$

$$\phi(t) = \left(u^T(t-1), \ldots, u^T(t-n)\right)^T$$

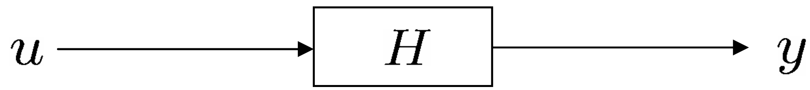
$$z(t) = H^T \phi(t)$$

$$z(t) z(t)^T = H^T \phi(t) \phi^T(t) H$$

$$\bar{E}\left(z(t) z(t)^T\right) = H^T \bar{E}\left(\phi(t) \phi^T(t)\right) H = H^T R_n H$$

equivalence established.

Spectrum of Filtered Signals



$$\Phi_y\left(e^{i\omega}\right) = H\left(e^{i\omega}\right)\Phi_u\left(e^{i\omega}\right)H^*\left(e^{i\omega}\right)$$

$$\overset{\text{SISO}}{=} \Phi_u\left(e^{i\omega}\right)\left|H\left(e^{i\omega}\right)\right|^2$$

If $u = WN$ signal $\phi_u = \lambda^2 I$

$$\Phi_y\left(e^{i\omega}\right) = \lambda^2 H H^*\left(e^{i\omega}\right)$$

$$\overset{\text{SISO}}{=} \lambda^2\left|H\left(e^{i\omega}\right)\right|^2$$

Generation of a process with a given Covariance:

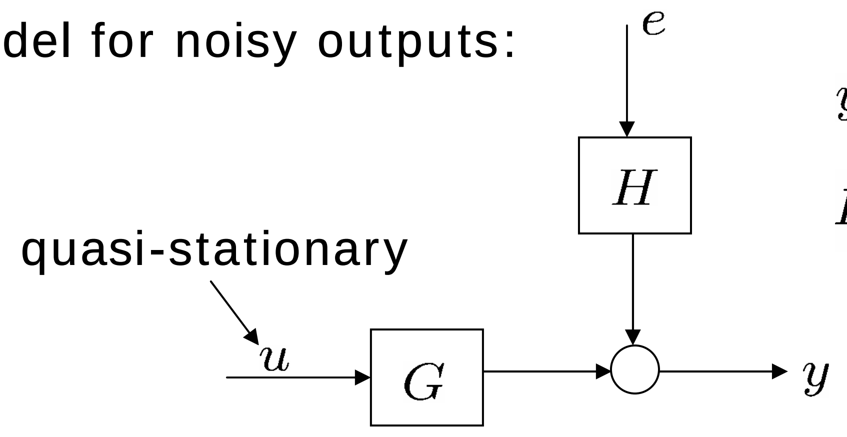
given $R_x(\tau)$, then x is the output of a filter with a WN signal as an input. The Filter is the spectral factor of $\Phi_x\left(e^{i\omega}\right)$;

$$\Phi_x\left(e^{i\omega}\right) = \underbrace{H\left(e^{i\omega}\right)} \cdot H^*\left(e^{i\omega}\right)$$

stable minimum phase.

Important relations

A model for noisy outputs:



$$y = Gu + He$$

$$E\left(e(t)e^T(t-\tau)\right) = \underbrace{\Lambda^2}_{\text{constant.}}\delta(\tau)$$

$$\Phi_y\left(e^{i\omega}\right) = G\Phi_u\left(e^{i\omega}\right)G^T\left(e^{i\omega}\right) + H\Lambda^2H^*\left(e^{i\omega}\right)$$

$$\Phi_{yu}\left(e^{i\omega}\right) = G\left(e^{i\omega}\right)\Phi_u(\omega)$$

- Very Important relations in system ID.
- Correlation methods are central in identifying an unknown plant.
- Proofs: Messy; Straight forward.

Ergodicity

- $x(t)$ is a stochastic process

Sample function or a realization

- Sample mean $= \bar{E}\tilde{x}(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \tilde{x}(t)$

- Sample Covariance $= \bar{E}\tilde{x}(t)\tilde{x}(t-\tau) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \tilde{x}(t)\tilde{x}(t-\tau)$

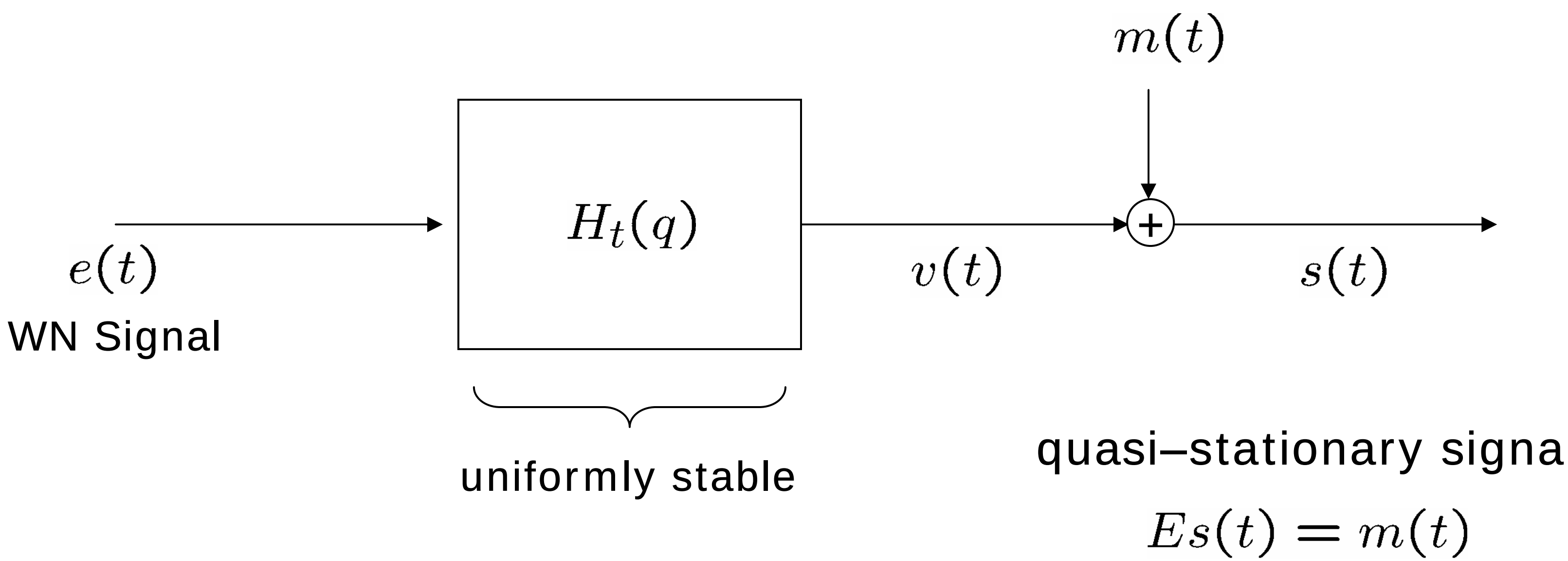
- A process is 2nd-order ergodic if

  mean $\triangleq Ex(t) =$ the sample mean of any realization.

  covariance $\triangleq Ex(t)x(t-\tau) =$ the sample covariance of any realization.

- Sample averages $\simeq$ Ensemble averages

A general ergodic process



$$\bar{E}s(t)s(t-\tau) = R_s(\tau) \qquad \text{w.p.1}$$

$$\frac{1}{N}\sum_{t=1}^{N}[s(t)m(t-\tau) - Es(t)m(t-\tau)] \to 0 \qquad \text{w.p.1}$$

$$\frac{1}{N}\sum_{t=1}^{N}[s(t)v(t-\tau) - Es(t)v(t-\tau)] \to 0 \qquad \text{w.p.1}$$

Remark:

Most of our computations will depend on a given realization of a quasi-stationary process. Ergodicity will allow us to make statements about repeated experiments.