6.453 Quantum Optical Communication
Spring 2009

Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

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Lecture Number 15
Fall 2008

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Date: Thursday, October 30, 2008

Continuous-variable teleportation.

Introduction

Today we'll develop the theory of continuous-variable teleportation, i.e., teleporting the quantum state of a single-mode electromagnetic field. Before delving into details, it's worth using what we have learned, from our treatment of qubit teleportation, to anticipate the features that we should expect from continuous-variable teleportation. First, Alice and Bob must share an entangled state, in this case a quadrature entangled state. Second, Alice must make a joint measurement on her electromagnetic field mode and that of Charlie, whose state is the state that is to be teleported. This measurement must do three things. First, it must not reveal any information about the states that Alice and Charlie held prior to the measurement. Second, it must contain all the information the Bob needs—beyond what is contained in his portion of the quadrature-entangled state that he shared with Alice—to replicate Charlie's state. Finally, it must not reveal any information to Bob about Charlie's state. In addition to these considerations, we must ensure that causality is not violated, i.e., the continuous-variable teleportation protocol cannot be run—from start to finish—at a rate that is faster than light speed.

The Teleportation Setup

Let us begin by reprising the description we presented at the end of Lecture 14. Slide 2 shows the entanglement generation setup on which continuous-variable teleportation relies. A two-mode parametric amplifier, with its input modes in their vacuum states, is governed by the two-mode Bogoliubov transformation

$$\hat{a}_{\text{out}_x} = \sqrt{G}\,\hat{a}_{\text{in}_x} + \sqrt{G-1}\,\hat{a}^\dagger_{\text{in}_y} \quad \text{and} \quad \hat{a}_{\text{out}_y} = \sqrt{G}\,\hat{a}_{\text{in}_y} + \sqrt{G-1}\,\hat{a}^\dagger_{\text{in}_x}, \tag{1}$$

where $G > 1$. The quadrature variances of the individual output modes are all super-shot noise, i.e.,

$$\langle \Delta\hat{a}^2_{\text{out}_{x k}}\rangle = \langle \Delta\hat{a}^2_{\text{out}_{y k}}\rangle = \frac{(2G-1)}{4} > \frac{1}{4}, \quad \text{for } k = 1, 2. \tag{2}$$

but the real and imaginary parts of the x- and y-polarized output modes are entangled, because

$$\left\langle\left(\frac{\Delta\hat{a}_{\text{out}_{x_1}} - \Delta\hat{a}_{\text{out}_{y_1}}}{\sqrt{2}}\right)^2\right\rangle = \left\langle\left(\frac{\Delta\hat{a}_{\text{out}_{x_2}} + \Delta\hat{a}_{\text{out}_{y_2}}}{\sqrt{2}}\right)^2\right\rangle \tag{3}$$

$$= \frac{(\sqrt{G}-\sqrt{G-1})^2}{4} \approx \frac{1}{16G} \ll \frac{1}{4}, \text{ for } G \gg 1. \tag{4}$$

This parametric amplifier is embedded in the continuous-variable teleportation system's transmitter (Alice) as shown on Slide 4, where, for brevity of notation, we have dropped the "out" designations on the parametric amplifier's output modes. Alice sends her $\hat{a}_x$ mode to Bob, through a (long-distance) lossy channel with transmissivity $0 < \gamma_x < 1$. She sends her $\hat{a}_y$ mode through a (short-distance) lossy channel[1] with transmissivity $0 < \gamma_y < 1$ to a 50/50 beam splitter, where it is combined with Charlie's $\hat{a}$ mode, whose state—$|\psi\rangle$, assumed to be pure—is to be teleported to Bob. The two outputs from this 50/50 beam splitter are then sent to balanced homodyne detection systems (built with quantum efficiency η photodetectors) that are set to measure the real and imaginary part quadratures of their illuminating fields. The classical outputs from these homodyne systems, denoted u and v, are sent to Bob over a classical communication channel, which is assumed to provide perfect (noiseless) transmission.[2]

Slide 5 shows the teleportation receiver (Bob). Bob starts with a strong coherent state field $|\sqrt{N_L}\rangle$, which he supplies to an electro-optic modulator driven by the classical information, u and v, that he received from Alice. The output of this modulator is combined—at an asymmetric beam splitter with transmissivity $T \approx 1$—with the field mode, $\hat{a}'_x$ that Bob received from Alice's lossy transmission of her $\hat{a}_x$ mode. The $\hat{a}_{\text{out}}$ mode emerging from this beam splitter then contains Bob's replica of Charlie's state.

The Transmitter Details

The field modes that enter the transmitter's 50/50 beam splitter shown on Slide 4 have annihilation operators $\hat{a}'_y$ and $\hat{a}$, where

$$\hat{a}'_y = \sqrt{\gamma_y}\,\hat{a}_y + \sqrt{1-\gamma_y}\,\hat{a}_{\gamma_y}, \tag{5}$$

[1]We should expect there to be loss on a long-distance channel. We are including loss in the short-distance channel because it will be purposefully employed by Alice to maximize the fidelity of the teleportation protocol, as we shall see later.

[2]Because this classical channel is light-speed limited, it alone precludes continuous-variable teleportation from violating causality. Note that u and v are *analog* quantities, i.e., they each take on a continuum of possible values. Thus, saying that Alice's classical communication link perfectly relays u and v to Bob is a much stronger assumption than the perfect classical communication assumption—of two bits from Alice to Bob—that we made in our treatment of the qubit teleportation protocol.

with $\hat{a}_{\gamma_y}$ being in its vacuum state and $\hat{a}$ being in state $|\psi\rangle$. The output modes from this beam splitter can then be taken to be $(\hat{a}+\hat{a}'_y)/\sqrt{2}$ and $(\hat{a}-\hat{a}'_y)/\sqrt{2}$, with these modes being the inputs, respectively, to the real-part and imaginary-part quadrature measurements that are performed by the two balanced homodyne systems. From our quantum theory of homodyne detection—with a normalization constant that differs from what we have previously employed by a factor of $\sqrt{2}$, and accounting for the sub-unity quantum efficiency—we have that the *classical* outcomes of the real and imaginary quadrature measurements have the following quantum measurement theory equivalents,

$$u \longleftrightarrow \hat{u} \;=\; \sqrt{\eta}\,(\hat{a}_1+\hat{a}'_{y_1})+\sqrt{2(1-\eta)}\,\hat{a}_{u_1} \tag{6}$$

$$v \longleftrightarrow \hat{v} \;=\; \sqrt{\eta}\,(\hat{a}_2-\hat{a}'_{y_2})+\sqrt{2(1-\eta)}\,\hat{a}_{v_2}, \tag{7}$$

where $\hat{a}_u$ and $\hat{a}_v$ are in their vacuum states.

It's worth examining the signal-to-noise ratios (SNRs) of the preceding measurements. Because $\hat{a}$ may be in an arbitrary state, so that its mean value $\langle\hat{a}\rangle$ might be zero, we shall take $\langle\hat{a}_1^2\rangle$ and $\langle\hat{a}_2^2\rangle$ as measures of the *squared* signal strengths in the real and imaginary quadratures of Charlie's state $|\psi\rangle$. Thus, for our SNR definitions we will use

$$\mathrm{SNR}_u \equiv \frac{\langle(\hat{u}|_{\text{due to }\hat{a}})^2\rangle}{\langle(\hat{u}|_{\textit{not}\text{ due to }\hat{a}})^2\rangle} \quad\text{and}\quad \mathrm{SNR}_v \equiv \frac{\langle(\hat{v}|_{\text{due to }\hat{a}})^2\rangle}{\langle(\hat{v}|_{\textit{not}\text{ due to }\hat{a}})^2\rangle}. \tag{8}$$

Because all the field modes that enter into $\hat{u}$ are in a product state, with all but the $\hat{a}$ mode definitely being in states with zero mean fields, we immediately find that

$$\langle\hat{u}^2\rangle = \underbrace{\eta\,\langle\hat{a}_1^2\rangle}_{\text{due to }\hat{a}} + \underbrace{\eta\,\langle\hat{a}'^2_{y_1}\rangle + 2(1-\eta)\,\langle\hat{a}^2_{u_1}\rangle}_{\textit{not}\text{ due to }\hat{a}}. \tag{9}$$

A similar calculation for $\hat{v}$ yields,

$$\langle\hat{v}^2\rangle = \underbrace{\eta\,\langle\hat{a}_2^2\rangle}_{\text{due to }\hat{a}} + \underbrace{\eta\,\langle\hat{a}'^2_{y_2}\rangle + 2(1-\eta)\,\langle\hat{a}^2_{v_2}\rangle}_{\textit{not}\text{ due to }\hat{a}}. \tag{10}$$

Next, we use

$$\langle\hat{a}'^2_{y_k}\rangle \;=\; \gamma_y\,\langle\hat{a}^2_{y_k}\rangle + (1-\gamma_y)\,\langle\hat{a}^2_{\gamma_{y_k}}\rangle \tag{11}$$

$$\;=\; \frac{\gamma_y(2G-1)}{4}+\frac{1-\gamma_y}{4}, \quad \text{for } k=1,2, \tag{12}$$

and

$$\langle\hat{a}^2_{u_1}\rangle = \langle\hat{a}^2_{v_2}\rangle = \frac{1}{4}, \tag{13}$$

to obtain the SNR expressions

$$\mathrm{SNR}_u \equiv \frac{\eta\langle\hat{a}_1^2\rangle}{\eta\langle\hat{a}'^2_{y_1}\rangle + 2(1-\eta)\langle\hat{a}^2_{u_1}\rangle} = \frac{4\eta\langle\hat{a}_1^2\rangle}{\eta[1+2\gamma_y(G-1)]+2(1-\eta)} \quad (14)$$

$$\mathrm{SNR}_v \equiv \frac{\eta\langle\hat{a}_2^2\rangle}{\eta\langle\hat{a}'^2_{y_2}\rangle + 2(1-\eta)\langle\hat{a}^2_{v_2}\rangle} = \frac{4\eta\langle\hat{a}_2^2\rangle}{\eta[1+2\gamma_y(G-1)]+2(1-\eta)}. \quad (15)$$

As $G \to \infty$ we get $\mathrm{SNR}_u \to 0$ and $\mathrm{SNR}_v \to 0$, but this should not dishearten us. The limit $G \to \infty$ gives maximum (perfect) quadrature entanglement, in that

$$\left\langle\left(\frac{\Delta\hat{a}_{\mathrm{out}_{x_1}} - \Delta\hat{a}_{\mathrm{out}_{y_1}}}{\sqrt{2}}\right)^2\right\rangle = \left\langle\left(\frac{\Delta\hat{a}_{\mathrm{out}_{x_2}} + \Delta\hat{a}_{\mathrm{out}_{y_2}}}{\sqrt{2}}\right)^2\right\rangle \quad (16)$$

$$= \frac{(\sqrt{G}-\sqrt{G-1})^2}{4} \to 0 \quad (17)$$

in this limit. We should expect that this perfect correlation between the real-part quadratures of the $\hat{a}_x$ and $\hat{a}_y$ modes, and the perfect anti-correlation between their imaginary-part quadratures will be essential to continuous-variable teleportation. Thus we shouldn't be surprised that $G \to \infty$ drives SNR_u and SNR_v to zero, because Alice cannot get any information about Charlie's state in the teleportation process.

For what follows, it will be useful to obtain an operator-equivalent for Alice's classical measurement data when that data is expressed as the complex number $u+jv$. We already know that

$$u + jv \longleftrightarrow \hat{u} + j\hat{v} = \sqrt{\eta}\,[(\hat{a}_1 + \hat{a}'_{y_1}) + j(\hat{a}_2 - \hat{a}'_{y_2})] + \sqrt{2(1-\eta)}\,(\hat{a}_{u_1} + j\hat{a}_{v_2}). \quad (18)$$

The first term on the right is easily seen to be $\sqrt{\eta}\,(\hat{a} + \hat{a}'^{\dagger}_y)$. Because the $\hat{a}_u$ and $\hat{a}_v$ modes are both in their vacuum states, the second term on the right in (18) is equivalent to $\sqrt{1-\eta}\,(\hat{a}_{\eta_u} + \hat{a}^\dagger_{\eta_v})$ where these annihilation operators represent fictitious modes that are also in their vacuum states. This substitution relies on two facts. The first is the commutator equivalence, i.e.,

$$[\hat{a}_{u_1} + j\hat{a}_{v_2}, \hat{a}_{u_1} - j\hat{a}_{v_2}] = [\hat{a}_{\eta_u} + \hat{a}^\dagger_{\eta_v}, \hat{a}^\dagger_{\eta_u} + \hat{a}_{\eta_v}] = 0. \quad (19)$$

The second is the statistical equivalence, i.e., we have

$$\langle e^{j\xi_1\sqrt{2(1-\eta)}\,\hat{a}_{u_1} + j\xi_2\sqrt{2(1-\eta)}\,\hat{a}_{v_2}}\rangle = \chi_W^{\rho_{a_u}}(\zeta^*,\zeta)\big|_{\zeta = j\xi_1\sqrt{\frac{1-\eta}{2}}}\,\chi_W^{\rho_{a_v}}(\zeta^*,\zeta)\big|_{\zeta=-\xi_2\sqrt{\frac{1-\eta}{2}}} \quad (20)$$

$$= e^{-(1-\eta)(\xi_1^2+\xi_2^2)/4}, \quad (21)$$

and

$$\langle e^{j\xi_1\sqrt{1-\eta}\,(\hat{a}_{\eta u_1} + \hat{a}_{\eta v_1}) + j\xi_2\sqrt{1-\eta}\,(\hat{a}_{\eta u_2} - \hat{a}_{\eta v_2})}\rangle$$

$$= \chi_W^{\rho_{a_{\eta u}}}(\zeta^*,\zeta)\big|_{\zeta=(j\xi_1-\xi_2)\frac{\sqrt{1-\eta}}{2}}\,\chi_W^{\rho_{a_{\eta v}}}(\zeta^*,\zeta)\big|_{\zeta=(j\xi_1+\xi_2)\frac{\sqrt{1-\eta}}{2}} = e^{-(1-\eta)(\xi_1^2+\xi_2^2)/4}. \quad (22)$$

Thus, in what follows, we will use[3]

$$\hat{u} + j\hat{v} = \sqrt{\eta}\,(\hat{a} + \hat{a}_y'^{\dagger}) + \sqrt{1-\eta}\,(\hat{a}_{\eta_u} + \hat{a}_{\eta_v}^{\dagger}). \tag{23}$$

The Receiver Details

Bob's receiver begins with the $\hat{a}_L$ mode in the strong coherent state $|\sqrt{N_L}\rangle$. This state is sufficiently strong that, when it is applied to the electro-optic modulator, the output mode, $\hat{a}_M$, can be taken to be in the coherent state $|K(u+jv)\rangle$, where $K > 0$ is a positive constant and u, v are Alice's classical measurement data. These measurement data are, of course, random, so what we are saying is that the output of Bob's modulator is the preceding coherent state *conditioned* on knowledge of u and v. Physically, for this to be so, is must be that for *all* values of u and v the electro-optic modulator acts as an attenuator, i.e., $|K(u+jv)| \le \sqrt{N_L}$. If this is not so, then Bob's receiver requires amplification, which we know will bring in additional quantum noise. So, strictly speaking, Bob will need $N_L \to \infty$ to keep his modulator in this attenuation regime.

Bob's output mode is given by

$$\hat{a}_{\text{out}} = \sqrt{T}\,\hat{a}_M - \sqrt{1-T}\,\hat{a}_x', \tag{24}$$

where

$$\hat{a}_x' = \sqrt{\gamma_x}\,\hat{a}_x + \sqrt{1-\gamma_x}\,\hat{a}_{\gamma_x} \tag{25}$$

is the mode that he received from Alice, with $\hat{a}_{\gamma_x}$ being in its vacuum state. Now, using $\hat{a} = \langle\hat{a}\rangle + \Delta\hat{a}$ for the preceding annihilation operators, and conditioning on knowledge of u and v, we get

$$\hat{a}_{\text{out}} = \sqrt{T}\,K(u+jv) + \sqrt{T}\,\Delta\hat{a}_M - \sqrt{(1-T)\gamma_x}\,\hat{a}_x - \sqrt{(1-T)(1-\gamma_x)}\,\hat{a}_{\gamma_x}, \tag{26}$$

with all the operators on the right-hand side being in their vacuum states, with the exception of $\hat{a}_x$, which is entangled with $\hat{a}_y$. Because we want $\hat{a}_{\text{out}}$ to be in a replica of Charlie's state $|\psi\rangle$, let us first try to match their mean fields. We have that

$$\langle\hat{a}_{\text{out}}\rangle = \sqrt{T}\,K\langle u+jv\rangle = \sqrt{T}\,K\langle\hat{u}+j\hat{v}\rangle = K\sqrt{\eta T}\,\langle\hat{a}\rangle, \tag{27}$$

so we will choose $K = 1/\sqrt{\eta T}$. Using this condition, and replacing $u+jv$ with $\hat{u}+j\hat{v}$ from (23), then gives us

$$\begin{aligned}\hat{a}_{\text{out}} &= \hat{a} + \sqrt{\gamma_y}\,\hat{a}_y^{\dagger} + \sqrt{1-\gamma_y}\,\hat{a}_{\gamma_y}^{\dagger} + \sqrt{\frac{1-\eta}{\eta}}\,(\hat{a}_{\eta_u} + \hat{a}_{\eta_v}^{\dagger}) + \sqrt{T}\,\Delta\hat{a}_M \\ &- \sqrt{(1-T)\gamma_x}\,\hat{a}_x - \sqrt{(1-T)(1-\gamma_x)}\,\hat{a}_{\gamma_x}. \end{aligned}\tag{28}$$

[3]Because $\hat{u}$ and $\hat{v}$ are operators that represent classical measurements u and v that were obtained *simultaneously*, it is nice to see that the right-hand side of the $\hat{u}+j\hat{v}$ expression commutes with its adjoint, as it must for $\hat{u}$ and $\hat{v}$ to be commuting observables.

All the terms on the right in (28) *after* $\hat{a}$ are noise terms. Of these, the most troublesome are those associated with $\hat{a}_x$ and $\hat{a}_y$. This is because these terms are (individually) in states with average photon numbers G with $G \gg 1$, whereas the other noise terms are all in their vacuum states. However, because of the quadrature entanglement between $\hat{a}_x$ and $\hat{a}_y$, *both* quadratures of $\hat{a}_x - \hat{a}_y^\dagger$ will have very low noise when $G \gg 1$. To reap this noise-cancellation benefit, we must balance the loss these modes suffer in the preceding $\hat{a}_{\text{out}}$ expression. Thus, we will take $\gamma \equiv \gamma_y = (1-T)\gamma_x$ to be the common value of this loss. Note that this is the purpose for our introduction of the loss modeled by γ_y inside Alice's transmitter, which might otherwise have been considered lossless. With this condition we reduce (28) to

$$\begin{aligned}\hat{a}_{\text{out}} &= \hat{a} - \sqrt{\gamma}\,(\hat{a}_x - \hat{a}_y^\dagger) + \sqrt{\frac{1-\eta}{\eta}}\,(\hat{a}_{\eta_u} + \hat{a}_{\eta_v}^\dagger) + \sqrt{T}\,\Delta\hat{a}_M \\ &+ \sqrt{1-\gamma}\,\hat{a}_{\gamma_y}^\dagger - \sqrt{(1-T)(1-\gamma_x)}\,\hat{a}_{\gamma_x}. \end{aligned} \tag{29}$$

There is yet one more transformation that we must make to further simplify our $\hat{a}_{\text{out}}$ result. Consider the following commutator,

$$\begin{aligned}&[\sqrt{T}\,\Delta\hat{a}_M - \sqrt{(1-T)(1-\gamma_x)}\,\hat{a}_{\gamma_x}, \sqrt{T}\,\Delta\hat{a}_M^\dagger - \sqrt{(1-T)(1-\gamma_x)}\,\hat{a}_{\gamma_x}^\dagger] \\ &= T + (1-T)(1-\gamma_x) = 1 - (1-T)\gamma_x = 1-\gamma. \end{aligned} \tag{30}$$

Because both $\Delta\hat{a}_M$ and $\hat{a}_{\gamma_x}$ are vacuum-state modes, the preceding commutator relation allows us to replace $\sqrt{T}\,\Delta\hat{a}_M - \sqrt{(1-T)(1-\gamma_x)}\,\hat{a}_{\gamma_x}$ with $\sqrt{1-\gamma}\,\hat{a}_N$, where $\hat{a}_N$ is the annihilation operator of a fictitious vacuum-state mode. We now have our final form for the $\hat{a}_{\text{out}}$ mode:

$$\hat{a}_{\text{out}} = \hat{a} + \sqrt{1-\gamma}\,(\hat{a}_N + \hat{a}_{\gamma_y}^\dagger) + \sqrt{\frac{1-\eta}{\eta}}\,(\hat{a}_{\eta_u} + \hat{a}_{\eta_v}^\dagger) - \sqrt{\gamma}\,(\hat{a}_x - \hat{a}_y^\dagger). \tag{31}$$

You should verify that all the terms in parentheses on the right-hand side are non-Hermitian operators that commute with their adjoints. As a result, the right-hand side does yield $[\hat{a}_{\text{out}}, \hat{a}_{\text{out}}^\dagger] = 1$, as it should. In the next section we will address the fidelity of the continuous-variable teleportation system whose operator-valued input-output relation is, as we have just shown, given by (31). To do that calculation, we will need to know the state of the $\hat{a}_{\text{out}}$ mode, as a function of the state of Charlie's input mode $\hat{a}$ and the parameters of the teleportation system.

In general, the state of the $\hat{a}_{\text{out}}$ mode will be mixed, and hence best represented as a density operator $\hat{\rho}_{\text{out}}$. We can use (31) to obtain the Wigner characteristic function for this density operator as follows:

$$\begin{aligned}
\chi_W^{\rho_{\rm out}}(\zeta^*\zeta) &= \chi_W^{\rho_a}(\zeta^*,\zeta)\chi_W^{\rho_{a_N}}(\sqrt{1-\gamma}\,\zeta^*,\sqrt{1-\gamma}\,\zeta)\\
&\times\ \chi_W^{\rho_{a_{\gamma y}}}(-\sqrt{1-\gamma}\,\zeta,-\sqrt{1-\gamma}\,\zeta^*)\\
&\times\ \chi_W^{\rho_{a_{\eta u}}}(\zeta^*\sqrt{(1-\eta)/\eta},\zeta\sqrt{(1-\eta)/\eta})\\
&\times\ \chi_W^{\rho_{a_{\eta v}}}(-\zeta\sqrt{(1-\eta)/\eta},-\zeta^*\sqrt{(1-\eta)/\eta})\\
&\times\ \chi_W^{\rho_{a_x,a_y}}(\zeta_x^*,\zeta_y^*,\zeta_x,\zeta_y)|_{\zeta_x=-\zeta\sqrt{\gamma},\zeta_y=-\zeta^*\sqrt{\gamma}}.
\end{aligned}\tag{32}$$

The vacuum-state contributions—from $\hat{a}_N$, $\hat{a}_{\gamma_y}$, $\hat{a}_{\eta_u}$, and $\hat{a}_{\eta_v}$ are easily accounted for, leading to

$$\begin{aligned}
\chi_W^{\rho_{\rm out}}(\zeta^*\zeta) &= \chi_W^{\rho_a}(\zeta^*,\zeta)\exp\left\{-\left[(1-\gamma)+\frac{1-\eta}{\eta}\right]|\zeta|^2\right\}\\
&\times\ \chi_W^{\rho_{a_x,a_y}}(\zeta_x^*,\zeta_y^*,\zeta_x,\zeta_y)|_{\zeta_x=-\zeta\sqrt{\gamma},\zeta_y=-\zeta^*\sqrt{\gamma}}.
\end{aligned}\tag{33}$$

From Lecture 13 we know that

$$\chi_W^{\rho_{a_x,a_y}}(\zeta_x^*,\zeta_y^*,\zeta_x,\zeta_y)=e^{-(|\zeta_x|^2+|\zeta_y|^2)(2G-1)/2+2{\rm Re}(\zeta_x\zeta_y)\sqrt{G(G-1)}},\tag{34}$$

which gives us

$$\chi_W^{\rho_{\rm out}}(\zeta^*\zeta)=\chi_W^{\rho_a}(\zeta^*,\zeta)\exp\left\{-\left[(1-\gamma)+\frac{1-\eta}{\eta}+s\gamma\right]|\zeta|^2\right\},\tag{35}$$

where

$$s\equiv(\sqrt{G}-\sqrt{G-1})^2\ll 1\quad\text{for } G\gg 1\tag{36}$$

is the squeezing factor.

Fidelity Analysis

The fidelity of the continuous-variable teleportation system, when Charlie's input state is $|\psi\rangle$, is defined to be

$$F\equiv\langle\psi|\hat{\rho}_{\rm out}|\psi\rangle.\tag{37}$$

In essence, this is the probability that the output state is $|\psi\rangle$. To find an expression for the fidelity, we will use (31) in conjunction with characteristic functions. From the homework we know that

$$\hat{\rho}_{\rm out}=\int\frac{{\rm d}^2\zeta}{\pi}\,\chi_A^{\rho_{\rm out}}(\zeta^*,\zeta)e^{-\zeta\hat{a}_{\rm out}^\dagger}e^{\zeta^*\hat{a}_{\rm out}}=\int\frac{{\rm d}^2\zeta}{\pi}\,\chi_W^{\rho_{\rm out}}(\zeta^*,\zeta)e^{-|\zeta|^2/2}e^{-\zeta\hat{a}^\dagger}e^{\zeta^*\hat{a}}.\tag{38}$$

We will content ourselves with finding the fidelity when Charlie's input state is the coherent state $|\alpha\rangle$. In this case we get

$$F = \langle\alpha|\hat{\rho}_{\text{out}}|\alpha\rangle = \int \frac{\mathrm{d}^2\zeta}{\pi}\, \chi_W^{\rho_{\text{out}}}(\zeta^*, \zeta) e^{-|\zeta|^2/2} e^{-\zeta\alpha^* + \zeta^*\alpha}. \tag{39}$$

From (35) with

$$\chi_W^{\rho_\alpha}(\zeta^*, \zeta) = e^{-\zeta^*\alpha + \zeta\alpha^* - |\zeta|^2/2} \tag{40}$$

we find that

$$F = \int \frac{\mathrm{d}^2\zeta}{\pi} \exp\left\{-\left[1 + (1-\gamma) + \frac{1-\eta}{\eta} + s\gamma\right]|\zeta|^2\right\} \tag{41}$$

$$= \frac{1}{1 + (1-\gamma) + \dfrac{1-\eta}{\eta} + s\gamma} \leq 1. \tag{42}$$

Equation (42) shows that to achieve $F \to 1$, we need $\eta \to 1$ (unity quantum efficiency photodetectors), $s \to 0$ (complete squeezing, which is equivalent to infinite parametric amplifier gain G), and no loss $\gamma \to 1$ (which requires $\gamma_x \to 1$ and $T \to 1$). These conditions are too restrictive to be hoped for even from idealized equipment. Thus, unlike qubit teleportation, for which the assumption of ideal equipment doesn't seem wildly impossible, such is not the case for continuous-variable teleportation.

The Road Ahead

Next time we will consider two approaches to quantum key distribution, which is a way to achieve completely secure communication through reliance on the laws of quantum mechanics.