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6.453 Quantum Optical Communication

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Reading: For the quantum theory of beam splitters:

- C.C. Gerry and P.L. Knight, *Introductory Quantum Optics* (Cambridge University Press, Cambridge, 2005) Sect. 6.2.

For the quantum theory of linear amplifiers:

- C.M. Caves, "Quantum limits on noise in linear amplifiers," Phys. Rev. D **26**, 1817–1839 (1982).
- L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, 1995), Chap. 20.
- H.A. Haus, *Electromagnetic Noise and Quantum Optical Measurements* (Springer Verlag, Berlin, 2000), Chap. 11.

Introduction

In this lecture we will begin by reprising the work done last time for the squeezed-state waveguide tap, focusing on the case in which the photodetectors used in the homodyne measurements at the output ports have quantum efficiencies that are less than unity. We will use this as a springboard from which to address the classical versus quantum theories for single-mode linear attenuation and single-mode linear amplification.

Optical Waveguide Tap with Ideal Photodetectors

Slide 3 reprises the quantum photodetection theory of the optical waveguide tap that was introduced in Lecture 1 and analyzed in Lecture 10. Assuming that the input signal is a coherent state $|a_{s_{\text{in}}}\rangle$ whose eigenvalue is real, and that the tap input mode is in a squeezed-vacuum state $|0;\mu,\nu\rangle$ with $\mu,\nu > 0$, we found that the signal input, signal output, and tap output signal-to-noise ratios were,

$$\mathrm{SNR_{in}} = 4|a_{s_{\rm in}}|^2 \tag{1}$$

$$\mathrm{SNR_{out}} = \frac{4T|a_{s_{\rm in}}|^2}{T+(1-T)(\mu-\nu)^2} \tag{2}$$

$$\mathrm{SNR_{tap}} = \frac{4(1-T)|a_{s_{\rm in}}|^2}{(1-T)+T(\mu-\nu)^2}, \tag{3}$$

so that for ν sufficiently large—i.e., when there is sufficient quadrature noise squeezing to make the tap input's noise contribution an insignificant component of the quadrature noise seen at the two output ports—we get

$$\mathrm{SNR_{out}} \approx \mathrm{SNR_{tap}} \approx \mathrm{SNR_{in}} = 4|a_{s_{\rm in}}|^2. \tag{4}$$

This result is beyond the realm of semiclassical photodetection, in that semiclassical photodetection predicts

$$\mathrm{SNR_{out}} + \mathrm{SNR_{tap}} = \mathrm{SNR_{in}}, \tag{5}$$

which is the performance that is obtained from the quantum theory when the tap input is in the vacuum state. The contrast between semiclassical and quantum behavior of the optical waveguide tap is illustrated on Slide 4, which compares the SNR tradeoffs—for the semiclassical (vacuum-state tap input) and squeezed-state (10 dB squeezed tap input)—as the tap transmissivity is varied from $T = 0$ to $T = 1$. Unfortunately, as we quickly showed in Lecture 10, sub-unity photodetector quantum efficiency can easily wash out the desirable non-classical behavior of the squeezed-state waveguide tap. Before quantifying that SNR-behavior loss, let us provide a more complete discussion of photodetection at sub-unity quantum efficiency.

Single-Mode Photodetection with $\eta < 1$ Photodetectors

Last time we introduced the following single-mode quantum photodetection model for a detector whose quantum efficiency, η, was less than one:

- Direct detection measures the number operator $\hat{N}' \equiv \hat{a}'^\dagger\hat{a}'$ associated with the photon annihilation operator

  $$\hat{a}' \equiv \sqrt{\eta}\,\hat{a} + \sqrt{1-\eta}\,\hat{a}_\eta, \tag{6}$$

  where $0 \le \eta < 1$ is the photodetector's quantum efficiency, $\hat{a}$ is the annihilation operator of the single-mode field that is illuminating the photodetector's light-sensitive region over the measurement interval, and $\hat{a}_\eta$ is the annihilation operator of a fictitious mode representing the loss associated with $\eta < 1$ prevailing. This fictitious mode is in its vacuum state, and its annihilation and creation operators commute with those associated with the illuminating field.

- Balanced homodyne detection measures the quadrature operator $\hat{a}'_\theta \equiv \text{Re}(\hat{a}'e^{-j\theta})$, where θ is the phase of the local oscillator relative to the signal mode.

- Balanced heterodyne detection realizes the probability operator-valued measurement associated with the annihilation operator $\hat{a}'$. Equivalently, balanced heterodyne detection can be said to provide a simultaneous measurement of the commuting observables that are the real and imaginary parts of $\sqrt{\eta}\,(\hat{a} + \hat{a}_I^\dagger) + \sqrt{1-\eta}\,(\hat{a}_\eta + \hat{a}_{I_\eta}^\dagger)$, where $\hat{a}_I$ is the annihilation operator of the image-band field that is illuminating the photodetector's light-sensitive region over the measurement interval,[1] and $\hat{a}_{I_\eta}$ is its associated $\eta < 1$ loss operator All four of the modal annihilation operators—$\hat{a}$, $\hat{a}_I$, $\hat{a}_\eta$, and $\hat{a}_{I_\eta}$—commute with each other and with each other's adjoint (creation) operator.

Last time we were not particularly explicit about the *semiclassical* theory for single-mode photodetection with quantum efficiency $\eta < 1$, so let us list its specifications now:

- Semiclassical direct detection—for a single-mode classical field with phasor a illuminating the photodetector's light-sensitive region over the measurement interval—yields a final count, N', that, conditioned on knowledge of a, is a Poisson-distributed random variable with mean $\eta|a|^2$, i.e.,

$$\Pr(\,N' = n \mid a = \alpha\,) = \frac{(\eta|\alpha|^2)^n e^{-\eta|\alpha|^2}}{n!}, \quad \text{for } n = 0, 1, 2, \ldots \tag{7}$$

- Semiclassical balanced homodyne detection—for a single-mode classical field with phasor a illuminating the photodetector's light-sensitive region over the measurement interval—produces a quadrature measurement outcome α'_θ that, conditioned on knowledge of a, is a variance-1/4 Gaussian random variable with mean value $\sqrt{\eta}\,a_\theta = \sqrt{\eta}\,\text{Re}(ae^{-j\theta})$

- Semiclassical balanced heterodyne detection—for a single-mode classical field with phasor a illuminating the photodetector's light-sensitive region over the measurement interval—yields quadrature measurement outcomes α'_1 and α'_2 that, conditioned on knowledge of a, are a pair of statistically independent variance-1/2 Gaussian random variables with mean values $\sqrt{\eta}\,a_1 = \sqrt{\eta}\,\text{Re}(a)$ and $\sqrt{\eta}\,a_2 = \sqrt{\eta}\,\text{Im}(a)$, respectively.

Because experiments invariably rely on photodetectors whose quantum efficiencies can, at best, *approach* unity quantum efficiency, we are interested in understanding the condition under which the measurement statistics obtained from single-mode

[1] Recall that for balanced heterodyne detection we have assumed that the excited signal mode has frequency ω and that the strong coherent state local oscillator has frequency $\omega - \omega_{\text{iF}}$. The image band field—which is in its vacuum state—then has frequency $\omega - 2\omega_{\text{IF}}$.

quantum photodetection coincide with those predicted by the single-mode semiclassical theory. It turns out that the answer is the same as for the case of unity quantum efficiency operation, i.e., if the $\hat{a}$ mode is in a coherent state or a classically-random mixture of such states—so that its density operator has a proper P-representation—then all the quantum photodetection statistics for an $\eta < 1$ detector are identical to their semiclassical counterparts. We'll see a proof of this statement in Lecture 12. For now, let's just present a few key results for direct detection and homodyne detection.

Suppose we perform direct detection on a quantum field using a sub-unity quantum efficiency detector. Then the mean of the measurement outcome N' obeys

$$\begin{aligned} \langle N' \rangle &= \langle \hat{a}'^\dagger \hat{a}' \rangle = \langle (\sqrt{\eta}\,\hat{a} + \sqrt{1-\eta}\,\hat{a}_\eta)^\dagger (\sqrt{\eta}\,\hat{a} + \sqrt{1-\eta}\,\hat{a}_\eta) \rangle && (8) \\ &= \eta \langle \hat{a}^\dagger \hat{a} \rangle + \sqrt{\eta(1-\eta)}\,(\langle \hat{a}^\dagger \hat{a}_\eta \rangle + \langle \hat{a}_\eta^\dagger \hat{a} \rangle) + (1-\eta)\langle \hat{a}_\eta^\dagger \hat{a}_\eta \rangle && (9) \\ &= \eta \langle \hat{a}^\dagger \hat{a} \rangle, && (10) \end{aligned}$$

where the last equality follows from $\hat{a}_\eta$ being in its vacuum state. A similar, but lengthier, calculation gives us

$$\begin{aligned} \langle N'^2 \rangle &= \langle (\hat{a}'^\dagger \hat{a}')^2 \rangle = \langle \hat{a}'^{\dagger 2} \hat{a}'^2 \rangle + \langle \hat{a}'^\dagger \hat{a}' \rangle && (11) \\ &= \eta^2 \langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle + \eta \langle \hat{a}^\dagger \hat{a} \rangle, && (12) \end{aligned}$$

where the second equality follows by squaring out and using $[\hat{a}', \hat{a}'^\dagger] = 1$, and the last equality follows from use of $\hat{a}' = \sqrt{\eta}\,\hat{a} + \sqrt{1-\eta}\,\hat{a}_\eta$ with $\hat{a}_\eta$ being in its vacuum state. From the preceding two results we have that photocount variance satisfies

$$\langle \Delta N'^2 \rangle = \eta \langle \hat{a}^\dagger \hat{a} \rangle + \eta^2 (\langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2). \qquad (13)$$

Two special cases of this variance formula are worth exhibiting. First, when the $\hat{a}$ mode is in the coherent state $|\alpha\rangle$ we find that

$$\langle \Delta N'^2 \rangle = \eta |\alpha|^2 = \langle N' \rangle, \qquad (14)$$

i.e., the photocount variance is Poissonian (equal to its mean), as expected from our statement that, even if $\eta < 1$ prevails, the quantum theory reduces to the semiclassical theory when the input mode is in a coherent state.

The second special case to examine is when the $\hat{a}$ mode is in the number state $|n\rangle$. Now we obtain

$$\langle N' \rangle = \eta n \quad \text{and} \quad \langle \Delta N'^2 \rangle = \eta n + \eta^2 [n(n-1) - n^2] = \eta(1-\eta) n. \qquad (15)$$

These results are consistent with the *binomial* distribution

$$\Pr(\, N' = m \mid \text{state } = |n\rangle \,) = \begin{pmatrix} n \\ m \end{pmatrix} \eta^m (1-\eta)^{(n-m)}, \quad \text{for } m = 0, 1, 2, \ldots, n, \qquad (16)$$

a result that you will derive on the homework. We can provide an intuitive explanation for this binomial distribution if we regard photons as rigid particles that, upon arrival at the photodetector, are successfully detected according to a Bernoulli process, i.e.,

- whether or not a particular photon registers a count is statistically independent of the count registration behavior of the other photons;
- the probability that a particular photon registers a count is η.

Appealing though this interpretation may be, it should not be relied on in general, as it does *not* give us a useful interpretation for homodyne detection with a sub-unity quantum efficiency photodetector.

Consider the case of balanced homodyne detection of the squeezed state $|\beta;\mu,\nu\rangle$ when $\theta = 0$, and β, μ, and ν real valued. In this case we find that the $\hat{a}_1'$ quadrature measurement has its mean and variance given by

$$\langle \hat{a}_1' \rangle = \sqrt{\eta}\,\beta(\mu-\nu) \quad \text{and} \quad \langle \Delta\hat{a}_1'^2 \rangle = \frac{\eta(\mu-\nu)^2 + (1-\eta)}{4}. \tag{17}$$

To get the complete statistics of this homodyne measurement, we can resort to characteristic functions. We know that

$$M_{a_1'}(jv) = \langle e^{-\zeta^*\hat{a}'+\zeta\hat{a}'^\dagger}\rangle|_{\zeta=jv/2} \tag{18}$$

$$= \left(\langle e^{-\zeta^*\sqrt{\eta}\,\hat{a}+\zeta\sqrt{\eta}\,\hat{a}^\dagger}\rangle\langle e^{-\zeta^*\sqrt{1-\eta}\,\hat{a}_\eta+\zeta\sqrt{1-\eta}\,\hat{a}_\eta^\dagger}\rangle\right)|_{\zeta=jv/2} \tag{19}$$

$$= M_{a_1}(jv\sqrt{\eta})\,M_{a_{\eta_1}}\!\left(jv\sqrt{1-\eta}\right), \tag{20}$$

where

$$M_{a_1}(jv) \equiv \langle e^{jv\mathrm{Re}(\hat{a})}\rangle \quad \text{and} \quad M_{a_{\eta_1}}(jv) \equiv \langle e^{jv\mathrm{Re}(\hat{a}_\eta)}\rangle. \tag{21}$$

Using our known quadrature statistics for the squeezed state and the vacuum state we then find

$$M_{a_1'}(jv) = e^{jv\sqrt{\eta}\,\beta(\mu-\nu)+v^2[\eta(\mu-\nu)^2+(1-\eta)]/8}, \tag{22}$$

which implies that the $\hat{a}_1'$ measurement outcome is a Gaussian random variable with mean and variance as given earlier. A simple derivation of this same result follows directly from

$$\hat{a}_1' = \sqrt{\eta}\,\hat{a}_1 + \sqrt{1-\eta}\,\hat{a}_{\eta_1}. \tag{23}$$

The signal and fictitious loss modes are in a product state, so the outcomes of measurements of their respective first quadratures are statistically independent random variables. For the states that we have assumed for these modes, the outcome of the $\hat{a}_1$ measurement is Gaussian distributed with mean $\beta(\mu-\nu)$ and variance $(\mu-\nu)^2/4$, and the outcome of the $\hat{a}_{\eta_1}$ measurement is Gaussian distributed with mean zero and variance 1/4. Because weighted sums of independent Gaussian random variables are Gaussian distributed, we are immediately led to the $\hat{a}_1'$ measurement statistics stated above and shown on Slide 6.

Optical Waveguide Tap with $\eta < 1$ Photodetectors

The homodyne detection results of the preceding section immediately afford what we need to perform the SNR evaluations for the squeezed-state waveguide tap when the *output* port homodyne measurements use quantum efficiency η detectors but the *input* port detection is still done at unity quantum efficiency. In general, we have that

$$\begin{aligned} \mathrm{SNR}_{\mathrm{in}} &= 4|a_{s_{\mathrm{in}}}|^2 && (24) \\ \mathrm{SNR}_{\mathrm{out}} &= \frac{4\eta T|a_{s_{\mathrm{in}}}|^2}{\eta T + (1-\eta) + 4\eta(1-T)\langle\Delta\hat{a}_{t_1}^2\rangle} && (25) \\ \mathrm{SNR}_{\mathrm{tap}} &= \frac{4\eta(1-T)|a_{s_{\mathrm{in}}}|^2}{\eta(1-T) + (1-\eta) + 4\eta T\langle\Delta\hat{a}_{t_1}^2\rangle}, && (26) \end{aligned}$$

where have continued to assume that the signal input's coherent-state eigenvalue is real and that the homodyne detectors are all measuring first (real part) quadratures. Slide 11 shows what transpires when $\eta = 0.7$ and we either use a vacuum-state tap input or a 10 dB squeezed tap input. In the latter case we still get performance that exceeds the former, but it barely crosses the "semiclassical frontier", $\mathrm{SNR}_{\mathrm{tap}}/\mathrm{SNR}_{\mathrm{in}} = 1 - \mathrm{SNR}_{\mathrm{out}}/\mathrm{SNR}_{\mathrm{in}}$. Physically, the degradation in our waveguide tap's performance comes from the zero-point fluctuations that are introduced by the $\hat{a}_\eta$ mode.

Single-Mode Classical Models for Linear Attenuation and Linear Amplification

Our study of sub-unity quantum efficiency is really an example of the quantum theory of linear attenuation (linear loss). Linear attenuation and linear amplification are important systems for us to understand, and so today we will begin their treatment—in the single-mode regime—by first considering their classical behavior.

For a single-mode classical field

$$E_{\mathrm{in}}(t) = \frac{a_{\mathrm{in}}e^{-j\omega t}}{\sqrt{T}}, \quad \text{for } 0 \le t \le T, \tag{27}$$

applied at the input to an ideal linear attenuator with transmissivity $0 \le L < 1$, we have that the resulting output field is

$$E_{\mathrm{out}}(t) = \frac{a_{\mathrm{out}}e^{-j\omega t}}{\sqrt{T}}, \quad \text{for } 0 \le t \le T, \tag{28}$$

where

$$a_{\mathrm{out}} = \sqrt{L}\,a_{\mathrm{in}}, \tag{29}$$

and, for simplicity, we have suppressed any propagation delay that the attenuator may impose. Equations (28) and (29) embody the linearity of this attenuator and the fact that classical physics does *not* require the inclusion of any noise.[2]

The classical theory for linear amplification of the single-mode field (27) takes a very similar form, viz., the output field is

$$E_{\rm out}(t) = \frac{a_{\rm out}e^{-j\omega t}}{\sqrt{T}}, \quad \text{for } 0 \le t \le T, \tag{30}$$

where

$$a_{\rm out} = \sqrt{G}\, a_{\rm in}, \tag{31}$$

with $1 < G < \infty$ being the amplifier's gain and once again propagation delay has been suppressed. Here too we have linearity without any noise injection.[3]

The absence of any noise injection in our classical models for single-mode linear attenuation and single-mode linear amplification leads to signal-to-noise preservation in classical measurements of energy and quadratures. In particular, suppose that we can perform *classical* measurements of $H_{\rm in} = \hbar\omega|a_{\rm in}|^2$ and $H_{\rm out} = \hbar\omega|a_{\rm out}|^2$, the energies at the inputs and outputs of the preceding attenuator and amplifier, where $a_{\rm in}$ is taken to be a complex-valued classical random variable. These are classical measurements, not semiclassical photodetection measurements, viz., we get to observe $H_{\rm in}$ and $H_{\rm out}$ *without* there being any shot noise. The only noise, therefore, on these measurements, is due to that which is intrinsic to $a_{\rm in}$. Thus, for the attenuator, we see that

$$\langle H_{\rm out}\rangle = L\hbar\omega\langle|a_{\rm in}|^2\rangle = L\langle H_{\rm in}\rangle, \tag{32}$$

and

$$\langle \Delta H_{\rm out}^2\rangle = (L\hbar\omega)^2\langle[\Delta(|a_{\rm in}|^2)]^2\rangle = L^2\langle\Delta H_{\rm in}^2\rangle, \tag{33}$$

so that there is no change in the signal-to-noise ratio

$$\text{SNR}_{H_{\rm out}} = \frac{\langle H_{\rm out}\rangle^2}{\langle\Delta H_{\rm out}^2\rangle} = \frac{\langle H_{\rm in}\rangle^2}{\langle\Delta H_{\rm in}^2\rangle} = \text{SNR}_{H_{\rm in}}. \tag{34}$$

If we perform the same analysis for the amplifier, we reproduce the preceding results with G appearing in place of L in (32) and (33), so that SNR preservation continues to hold.

Now suppose that we perform *classical* measurements of the input and output quadrature components $a_{{\rm in}_\theta} = \text{Re}(a_{\rm in}e^{-j\theta})$ and $a_{{\rm out}_\theta} = \text{Re}(a_{\rm out}e^{-j\theta})$ for the amplifier. Here we find that

$$\langle a_{{\rm out}_\theta}\rangle = \sqrt{G}\,\langle a_{{\rm in}_\theta}\rangle \quad \text{and} \quad \langle\Delta a_{{\rm out}_\theta}^2\rangle = G\langle\Delta a_{{\rm in}_\theta}^2\rangle, \tag{35}$$

[2]Here we are neglecting the thermal noise associated with dissipative elements in statistical mechanics.

[3]Experimenters are well aware that real amplifiers have noise figures that measure the extra noise that they inject. Once again, however, the fundamental requirement for there to be such noise arises out of statistical mechanical considerations that we are omitting from our classical theory.

and once again signal-to-noise ratio is preserved,

$$\text{SNR}_{a_{\text{out}_\theta}} = \frac{\langle a_{\text{out}_\theta}\rangle^2}{\langle \Delta a^2_{\text{out}_\theta}\rangle} = \frac{\langle a_{\text{in}_\theta}\rangle^2}{\langle \Delta a^2_{\text{in}_\theta}\rangle} = \text{SNR}_{a_{\text{in}_\theta}}. \tag{36}$$

If we perform the same analysis for the attenuator, we reproduce these quadrature measurement results with L appearing in place of G in (35), so that SNR preservation continues to hold.

Single-Mode Quantum Models for Linear Attenuation and Linear Amplification

In Lecture 10 we went from the classical input-output relation for a transmissivity-T beam splitter to its quantum version by the simple artifice of changing every classical phasor a to a corresponding annihilation operator $\hat{a}$. This was in keeping with our work on the quantum harmonic oscillator in that the classical field (27) at the input to our linear attenuator or linear amplifier becomes the field operator,

$$\hat{E}_{\text{in}}(t) = \frac{\hat{a}_{\text{in}} e^{-j\omega t}}{\sqrt{T}}, \quad \text{for } 0 \le t \le T, \tag{37}$$

with $\hat{a}$ being an annihilation operator, in the quantum theory. So, can't we just write

$$\hat{E}_{\text{out}}(t) = \frac{\hat{a}_{\text{out}} e^{-j\omega t}}{\sqrt{T}}, \quad \text{for } 0 \le t \le T, \tag{38}$$

for the output field operator—with propagation delay suppressed—and use

$$\hat{a}_{\text{out}} = \sqrt{L}\,\hat{a}_{\text{in}} \quad \text{and} \quad \hat{a}_{\text{out}} = \sqrt{G}\,\hat{a}_{\text{in}}, \tag{39}$$

for the respective output annihilation operators of the linear attenuator and the linear amplifier? The answer is quite definitely NO! Both the input *and* the output annihilation operators *must* have the canonical commutator relations with their associated creation operators, i.e., we require

$$[\hat{a}_{\text{out}}, \hat{a}^\dagger_{\text{out}}] = [\hat{a}_{\text{in}}, \hat{a}^\dagger_{\text{in}}] = 1, \tag{40}$$

in order to preserve the Heisenberg uncertainty principle for quadrature measurements. The naïve quantum input-output relations (39) yield

$$[\hat{a}_{\text{out}}, \hat{a}^\dagger_{\text{out}}] = \begin{cases} L[\hat{a}_{\text{in}}, \hat{a}^\dagger_{\text{in}}] = L < 1, & \text{for the attenuator} \\ G[\hat{a}_{\text{in}}, \hat{a}^\dagger_{\text{in}}] = G > 1, & \text{for the amplifier,} \end{cases} \tag{41}$$

so neither one is acceptable as neither one gives the correct uncertainty principle for a single-mode field,

$$\langle \Delta \hat{a}^2_{\text{out}_1}\rangle \langle \Delta \hat{a}^2_{\text{out}_2}\rangle \ge 1/16. \tag{42}$$

It turns out that we already now how to "fix" the quantum input-output relation for linear attenuation. What we need is

$$\hat{a}_{\text{out}} = \sqrt{L}\,\hat{a}_{\text{in}} + \sqrt{1-L}\,\hat{a}_L, \tag{43}$$

where $\hat{a}_L$ is the annihilation operator of an auxiliary mode associated with the attenuation process. This, after all, is just what we did for sub-unity quantum efficiency, hence we know that it gives $[\hat{a}_{\text{out}}, \hat{a}^\dagger_{\text{out}}] = 1$. Furthermore, our understanding of beam splitters gives us a physical interpretation of this input-output relation. We can regard $\hat{a}_{\text{in}}$ and $\hat{a}_L$ as the annihilation operators for single-mode fields that illuminate the two input ports of a transmissivity-L beam splitter, and take $\hat{a}_{\text{out}}$ to be the annihilation operator of the output port from that beam splitter which is coupled to the signal ($\hat{a}_{\text{in}}$) port by the transmissivity-L path. Ordinarily, the auxiliary mode will be in its vacuum state, as we have stated for the case of sub-unity quantum efficiency photodetection, but this need not be the case. The auxiliary mode might be coupled to a thermal bath, so that its state is given by the density operator

$$\hat{\rho}_{a_L} = \sum_{n=0}^{\infty} \frac{\bar{N}^n}{(\bar{N}+1)^{n+1}}\,|n\rangle_{LL}\langle n|, \tag{44}$$

where $\{|n\rangle_L\}$ are the $\hat{a}_L$ mode's numbers states and

$$\bar{N} \equiv \frac{1}{e^{\hbar\omega/k_B T_R} - 1}, \tag{45}$$

with k_B being Boltzmann's constant and T_R being the reservoir's absolute temperature.[4] Commutator preservation—and hence Heisenberg uncertainty principle preservation—is unaffected by the choice of state for the $\hat{a}_L$ mode.

For both the waveguide tap and free-space diffraction it is easy to see why a beam splitter relation governs the attenuation process, and the physical locus of the auxiliary mode can be readily identified. For sub-unity quantum efficiency photodetection, we can see why a beam splitter relation makes phenomenological sense, but we may not be able to identify a precise physical locus for the auxiliary mode that enters into the quantum input-output relation $\hat{a}' = \sqrt{\eta}\,\hat{a} + \sqrt{1-\eta}\,\hat{a}_\eta$. The situation for the linear amplifier—at least given what we have learned so far—falls very much into this latter category, i.e., we can (and will) say that the appropriate single-mode input-output relation for the linear amplifier is

$$\hat{a}_{\text{out}} = \sqrt{G}\,\hat{a}_{\text{in}} + \sqrt{G-1}\,\hat{a}^\dagger_G, \tag{46}$$

where $\hat{a}_G$ is the annihilation operator of an auxiliary mode associated with the amplification process, but we do not have (as yet) any example with which to identify a

[4] On the homework you will see that this state is a maximum entropy state for the given average photon number.

physical locus for this auxiliary mode. It turns out—although we will not show this to be so—that if the amplifier is a laser gain cell (a laser operated below its oscillation threshold), then the $\hat{a}_G$ mode is associated with spontaneous emission. We will see—later this term—that if the amplifier is a two-mode parametric amplifier, then the $\hat{a}_G$ mode is associated with the parametric amplifier's idler mode. For now, let's content ourselves with verifying that (46) preserves commutator brackets. We have that

$$[\hat{a}_{\rm out}, \hat{a}_{\rm out}^\dagger] = G[\hat{a}_{\rm in}, \hat{a}_{\rm in}^\dagger] + (G-1)[\hat{a}_G^\dagger, \hat{a}_G] = G - (G-1) = 1, \tag{47}$$

where the first equality follows because $\hat{a}_{\rm in}$ and $\hat{a}_G$ are associated with different field modes, and the second equality uses the commutator relation for annihilation operators and their associated creation operators. The least extra noise that the $\hat{a}_G^\dagger$ term can inject into the $\hat{a}_{\rm out}$ mode is when the former is in its vacuum state. As was the case for the linear attenuator, however, there are physical examples in which $\hat{a}_G$ may be in a non-vacuum state owing, e.g., to its being coupled to a thermal reservoir. Nevertheless, commutator preservation—and hence Heisenberg uncertainty principle preservation—is unaffected by the choice of state for the $\hat{a}_G$ mode.

To contrast our quantum input-output equations with their classical counterparts, we shall consider the relations between input and output SNRs for direct detection and homodyne detection arising from the quantum theory. We'll do the homodyne (quadrature measurement) case first, because it is simpler and, then we will turn to the direct detection case.

Homodyne Detection SNR Relations

For the attenuator, with the $\hat{a}_L$ mode in its vacuum state, we have that

$$\langle \hat{a}_{{\rm out}_\theta} \rangle = \sqrt{L}\,\langle \hat{a}_{{\rm in}_\theta} \rangle + \sqrt{1-L}\,\langle \hat{a}_{L_\theta} \rangle = \sqrt{L}\,\langle \hat{a}_{{\rm in}_\theta} \rangle, \tag{48}$$

and

$$\langle \Delta\hat{a}_{{\rm out}_\theta}^2 \rangle = L\langle \Delta\hat{a}_{{\rm in}_\theta}^2 \rangle + (1-L)\langle \Delta\hat{a}_{L_\theta}^2 \rangle = L\langle \Delta\hat{a}_{{\rm in}_\theta}^2 \rangle + (1-L)/4, \tag{49}$$

where we have used the fact that the input and auxiliary modes are in a product state to obtain the first equality. The output SNR is now *degraded* from the input SNR,

$${\rm SNR}_{{\rm out}_\theta} = \frac{L\langle \hat{a}_{{\rm in}_\theta} \rangle^2}{L\langle \Delta\hat{a}_{{\rm in}_\theta}^2 \rangle + (1-L)/4} < \frac{\langle \hat{a}_{{\rm in}_\theta} \rangle^2}{\langle \Delta\hat{a}_{{\rm in}_\theta}^2 \rangle} = {\rm SNR}_{{\rm in}_\theta}, \tag{50}$$

because of the zero-point fluctuations contributed by the $\hat{a}_L$ mode. In the special case of a coherent-state input mode, the preceding variance and SNR results for the attenuator become

$$\langle \Delta\hat{a}_{{\rm out}_\theta}^2 \rangle = 1/4 \quad \text{and} \quad {\rm SNR}_{{\rm out}_\theta} = L{\rm SNR}_{{\rm in}_\theta}. \tag{51}$$

Here we see that attenuation of coherent state light leads to a linear degradation of the homodyne detection SNR.

For the amplifier, with the $\hat{a}_G$ mode in its vacuum state, we find that

$$\hat{a}_{\text{out}_1} = \sqrt{G}\,\hat{a}_{\text{in}_1} + \sqrt{G-1}\,\hat{a}_{G_1} \quad \text{and} \quad \hat{a}_{\text{out}_2} = \sqrt{G}\,\hat{a}_{\text{in}_2} - \sqrt{G-1}\,\hat{a}_{G_2}, \qquad (52)$$

from which calculations similar to what we just did for the attenuator—whose details will be left as exercises for the reader—lead to

$$\langle \hat{a}_{\text{out}_k}\rangle = \sqrt{G}\,\langle \hat{a}_{\text{in}_k}\rangle \quad \text{and} \quad \langle \Delta\hat{a}^2_{\text{out}_k}\rangle = G\langle \Delta\hat{a}^2_{\text{in}_k}\rangle + (G-1)/4, \qquad (53)$$

for $k = 1, 2$, leading to the output signal-to-noise ratio expression

$$\text{SNR}_{\text{out}_k} = \frac{G\langle \hat{a}_{\text{in}_k}\rangle^2}{G\langle \Delta\hat{a}^2_{\text{in}_k}\rangle + (G-1)/4} < \frac{\langle \hat{a}_{\text{in}_k}\rangle^2}{\langle \Delta\hat{a}^2_{\text{in}_k}\rangle} = \text{SNR}_{\text{in}_k}, \quad \text{for } k = 1, 2, \qquad (54)$$

where the degradation in the output SNR is due to the zero-point fluctuations contributed by the $\hat{a}_G$ mode. In the special case of a coherent-state input mode, the preceding variance and SNR results for the amplifier become

$$\langle \Delta\hat{a}^2_{\text{out}_k}\rangle = (2G-1)/4 \quad \text{and} \quad \text{SNR}_{\text{out}_k} = \frac{G}{2G-1}\text{SNR}_{\text{in}_k} \quad \text{for } k = 1, 2. \qquad (55)$$

Here we see that amplification of coherent state light leads to a degradation of the homodyne detection SNR that is at most 3 dB, with maximum degradation occurring in the limit $G \to \infty$.

Direct Detection SNR Relations

For the attenuator—with the $\hat{a}_L$ mode in its vacuum state and the $\hat{a}_{\text{in}}$ mode in the coherent state $|\alpha_{\text{in}}\rangle$—there is no work to be done, because this case coincides with that of sub-unity quantum efficiency photodetection. We immediately have

$$\langle \hat{N}_{\text{out}}\rangle = L\langle \hat{N}_{\text{in}}\rangle \quad \text{and} \quad \langle \Delta\hat{N}^2_{\text{out}}\rangle = L\langle \hat{N}_{\text{in}}\rangle. \qquad (56)$$

Thus the photocount variance is Poissonian,[5] and the SNR degrades linearly with decreasing L,

$$\text{SNR}_{N_{\text{out}}} = \frac{\langle \hat{N}_{\text{out}}\rangle^2}{\langle \Delta\hat{N}^2_{\text{out}}\rangle} = \frac{L\langle \hat{N}_{\text{in}}\rangle^2}{\langle \Delta\hat{N}^2_{\text{in}}\rangle} = L\text{SNR}_{N_{\text{in}}}. \qquad (57)$$

For the amplifier, however, there is work to be done. Here, assuming that the $\hat{a}_G$ mode is in its vacuum state and the $\hat{a}_{\text{in}}$ mode is in the coherent state $|\alpha_{\text{in}}\rangle$, we can easily find the average photon count at the amplifier's output:

$$\langle \hat{N}_{\text{out}}\rangle = \langle(\sqrt{G}\,\hat{a}_{\text{in}} + \sqrt{G-1}\,\hat{a}^\dagger_G)^\dagger(\sqrt{G}\,\hat{a}_{\text{in}} + \sqrt{G-1}\,\hat{a}^\dagger_G)\rangle \qquad (58)$$

$$= G\langle \hat{a}^\dagger_{\text{in}}\hat{a}_{\text{in}}\rangle + (G-1)\langle \hat{a}_G\hat{a}^\dagger_G\rangle = G|\alpha_{\text{in}}|^2 + G - 1 = G\langle \hat{N}_{\text{in}}\rangle + G - 1. \qquad (59)$$

[5]Indeed, as we have seen earlier the photocount is Poisson distributed in this case.

The variance is harder to compute. We start by using commutator brackets to obtain

$$\langle \Delta \hat{N}_{\text{out}}^2 \rangle = \langle \hat{a}_{\text{out}}^\dagger \hat{a}_{\text{out}} \rangle + (\langle \hat{a}_{\text{out}}^{\dagger 2} \hat{a}_{\text{out}}^2 \rangle - \langle \hat{a}_{\text{out}}^\dagger \hat{a}_{\text{out}} \rangle^2) \tag{60}$$

Next, we work on $\langle \hat{a}_{\text{out}}^{\dagger 2} \hat{a}_{\text{out}}^2 \rangle$ by writing $\hat{a}_{\text{out}}$ in terms of $\hat{a}_{\text{in}}$ and $\hat{a}_G^\dagger$, multiplying out, and using the assumed coherent states to evaluate the averages. The details are left to the reader, but the final result is

$$\langle \Delta \hat{N}_{\text{out}}^2 \rangle = [G\langle \hat{N}_{\text{in}} \rangle + (G-1)] + [2G(G-1)\langle \hat{N}_{\text{in}} \rangle + (G-1)^2]. \tag{61}$$

In Lecture 12 we shall rederive the photocount variance by a different route, and in so doing develop physical interpretations for the two bracketed terms in this $\langle \Delta \hat{N}_{\text{out}}^2 \rangle$ expression. For now, we just note that[6]

$$\text{SNR}_{N_{\text{out}}} = \frac{\langle \hat{N}_{\text{out}} \rangle^2}{\langle \Delta \hat{N}_{\text{out}}^2 \rangle} < \frac{\langle \hat{N}_{\text{in}} \rangle^2}{\langle \Delta \hat{N}_{\text{in}}^2 \rangle} = \text{SNR}_{N_{\text{in}}}. \tag{62}$$

The Road Ahead

In the next lecture we shall continue our work on linear attenuation and linear amplification. We will use characteristic functions to obtain their complete statistical characterizations. We shall also introduce the two-mode description of parametric amplification, which will lead us to distinguish between phase-*insensitive* linear amplification—what we have discussed today—and phase-*sensitive* linear amplification.

[6]Because $\langle \hat{N}_{\text{out}} \rangle$ includes a $G-1$ term that has no relation to the input mean $\langle \hat{N}_{\text{in}} \rangle$, a more appropriate definition of the output signal-to-noise ratio is $\text{SNR}_{N_{\text{out}}} = (G\langle \hat{N}_{\text{in}} \rangle)^2 / \langle \Delta \hat{N}_{\text{out}}^2 \rangle$, i.e., only count the portion of the output mean that is proportional to the input mean as "signal," but include all contributions to the variance of the output as "noise." With this definition we still find $\text{SNR}_{\text{out}} < \text{SNR}_{\text{in}}$. Moreover, when $\langle \hat{N}_{\text{in}} \rangle \gg 1$ we find that this new definition leads to $\lim_{G\to\infty} \text{SNR}_{\text{out}} = \text{SNR}_{\text{in}}/2$.