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Single-mode signatures of non-classical light, and the squeezed-state waveguide tap

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Introduction

Today's lecture has two purposes. First, we will summarize our results on single-mode semiclassical and quantum photodetection theory by focusing on the non-classical signatures that can be seen in direct, homodyne, and heterodyne detection. This will include showing how a set of measurements on an ensemble of identically prepared quantum harmonic oscillators can be used to infer their state. Our second task, in today's lecture, will be to revisit the squeezed-state waveguide tap that was described in Lecture 1. Armed with our understanding of single-mode photodetection, we will be able to derive the signal-to-noise ratio results that were presented in that opening lecture. Furthermore, we shall take this opportunity to extend our single-mode photodetection theory to include photodetectors whose quantum efficiencies are less than unity.

Semiclassical versus Quantum Photodetection

Slide 3 summarizes what we have learned about semiclassical photodetection of a single-mode field. Here, to simplify the notation, we have suppressed the spatial dependence that we included in last lecture, so that $\hbar\omega \int_0^T \mathrm{d}t\,|E(t)|^2$ is now the energy illuminating the photodetector's light-sensitive region during the measurement interval $0 \le t \le T$. Direct detection yields a final count N that, conditioned on knowledge of the phasor a, is a Poisson-distributed random variable with mean $|a|^2$. Homodyne detection gives a quadrature value α_θ that, conditioned on knowledge of a, is a variance-1/4 Gaussian-distributed random variable with mean value $a_\theta = \mathrm{Re}(ae^{-j\theta})$. Here, θ is the phase shift of the strong local oscillator relative to the signal, i.e., $E_{\mathrm{LO}}(t) = \sqrt{N_{\mathrm{LO}}}\, e^{-j(\omega t-\theta)}/\sqrt{T}$ for $0 \le t \le T$, with $N_{\mathrm{LO}} \to \infty$. Heterodyne detection yields a pair of quadrature values, α_1 and α_2, that, conditioned on knowledge of a, are statistically independent, variance-1/2, Gaussian-distributed random variables with mean values $a_1 = \mathrm{Re}(a)$ and $a_2 = \mathrm{Im}(a)$, respectively.

Slide 4 summarizes what we have learned about quantum photodetection of a single-mode field. Here too we have simplified our notation by suppressing the spatial dependence that we included last time. Thus, $\hat{a}$ is still a photon annihilation operator, so that $\hbar\omega \int_0^T \mathrm{d}t\, \hat{E}^\dagger(t)\hat{E}(t)$ is still the observable representing the total energy (above the zero-point energy) illuminating the photodetector's light-sensitive region during the measurement interval $0 \le t \le T$. Direct detection yields a final count N that is equivalent to the quantum measurement of $\hat{N} \equiv \hat{a}^\dagger\hat{a}$, the number operator associated with the excited mode.[1] Homodyne detection gives a quadrature value α_θ that is equivalent to the quantum measurement of $\hat{a}_\theta = \mathrm{Re}(\hat{a}e^{-j\theta})$, where θ is again the phase shift of the strong local oscillator relative to the signal. Heterodyne detection yields a pair of quadrature values, α_1 and α_2, such that $\alpha = \alpha_1 + j\alpha_2$ is equivalent to the positive operator-valued measurement of $\hat{a}$. Alternatively, we can say that α is equivalent to the quantum measurement of $\hat{a} + \hat{a}_I^\dagger$. Here, $\hat{a}_I$ is the annihilation operator of the unexcited (vacuum-state) image mode $\hat{a}_I e^{-j(\omega-2\omega_{\mathrm{IF}})t}/\sqrt{T}$ for $0 \le t \le T$. Because $[\hat{a}+\hat{a}_I^\dagger, \hat{a}^\dagger + \hat{a}_I] = 0$, the real and imaginary parts of $\hat{a}+\hat{a}_I^\dagger$ are commuting observables. Thus they can be measured simultaneously, and their measurements are equivalent to the heterodyne outputs α_1 and α_2, respectively, when the image mode is in its vacuum state.

Non-classical Signatures in Photodetection Variances

Three of the most important non-classical signatures—ways in which quantum photodetection theory makes predictions that are impossible to reproduce from semiclassical photodetection theory—appear in the variances of direct, homodyne, and heterodyne detection. Because it is very difficult, experimentally, to produce an optical field that is completely deterministic (in classical electromagnetism) or in a pure state (quantum mechanically) it is important for us to make these variance comparisons when the classical field on Slide 3 is allowed to have a be a complex-valued random variable whose joint probability density, for its real and imaginary parts, is $p_a(\alpha)$. LIkewise, we will take the quantum field on Slide 4 to be in a mixed state characterized by the density operator $\hat{\rho}_a$ for its single excited mode. We now need to perform a little exercise in iterated expectation before we can compare and contrast the semiclassical and quantum photodetection variances of direct, homodyne, and heterodyne detection.

Consider semiclassical direct detection when a is a complex-valued random variable with pdf $p_a(\alpha)$. We then have that

$$\Pr(N = n) = \int \mathrm{d}^2\alpha\, p_a(\alpha) \Pr(\, N = n \mid a = \alpha\,) \tag{1}$$

$$= \int \mathrm{d}^2\alpha\, p_a(\alpha) \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!}, \text{ for } n = 0, 1, 2, \ldots \tag{2}$$

[1] Recall, from last lecture, that, in general, the final count N is equivalent to measurement of the total photon number observable, $\hat{N}_T = \int_0^T \mathrm{d}t\, \hat{E}^\dagger(t)\hat{E}(t)$. When the single mode shown on Slide 4 is the only non-vacuum mode in the field, this reduces to $\hat{N}_T = \hat{a}^\dagger\hat{a}$.

From this it follows that the mean, mean-square, and variance of N are:

$$\begin{aligned}
\langle N \rangle &= \sum_{n=0}^{\infty} n \Pr(N=n) = \int \mathrm{d}^2\alpha\, p_a(\alpha) \sum_{n=0}^{\infty} n \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!} \qquad (3) \\
&= \int \mathrm{d}^2\alpha\, |\alpha|^2 p_a(\alpha) = \langle |a|^2 \rangle, \qquad (4)
\end{aligned}$$

$$\begin{aligned}
\langle N^2 \rangle &= \sum_{n=0}^{\infty} n^2 \Pr(N=n) = \int \mathrm{d}^2\alpha\, p_a(\alpha) \sum_{n=0}^{\infty} n^2 \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!} \qquad (5) \\
&= \int \mathrm{d}^2\alpha\, (|\alpha|^2 + |\alpha|^4) p_a(\alpha) = \langle |a|^2 \rangle + \langle |a|^4 \rangle = \langle N \rangle + \langle |a|^4 \rangle, \qquad (6)
\end{aligned}$$

and

$$\langle \Delta N^2 \rangle = \langle N^2 \rangle - \langle N \rangle^2 = \langle N \rangle + (\langle |a|^4 \rangle - \langle |a|^2 \rangle^2) = \langle N \rangle + \mathrm{var}(|a|^2). \qquad (7)$$

Now consider semiclassical homodyne detection when a is a complex-valued random variable with pdf $p_a(\alpha)$. In this case, because α_θ is Gaussian with mean a_θ and variance $1/4$ when a_θ is known, we find the following results for the mean, mean-square, and variance of α_θ:

$$\begin{aligned}
\langle \alpha_\theta \rangle &= \int \mathrm{d}^2\beta\, p_a(\beta) \beta_\theta = \langle a_\theta \rangle \qquad (8) \\
\langle \alpha_\theta^2 \rangle &= \int \mathrm{d}^2\beta\, p_a(\beta)(\beta_\theta^2 + 1/4) = \langle a_\theta^2 \rangle + 1/4 \qquad (9) \\
\langle \Delta\alpha_\theta^2 \rangle &= \langle \alpha_\theta^2 \rangle - \langle \alpha_\theta \rangle^2 = \langle \Delta a_\theta^2 \rangle + 1/4. \qquad (10)
\end{aligned}$$

Finally, consider semiclassical heterodyne detection when a is a complex-valued random variable with pdf $p_a(\alpha)$. Here, using the conditional statistics given earlier, the mean, mean-square, and variance of the quadratures, α_k for $k = 1, 2$, are found to be:

$$\begin{aligned}
\langle \alpha_k \rangle &= \int \mathrm{d}^2\beta\, p_a(\beta) \beta_k = \langle a_k \rangle \qquad (11) \\
\langle \alpha_k^2 \rangle &= \int \mathrm{d}^2\beta\, p_a(\beta)(\beta_k^2 + 1/2) = \langle a_k^2 \rangle + 1/2 \qquad (12) \\
\langle \Delta\alpha_k^2 \rangle &= \langle \alpha_k^2 \rangle - \langle \alpha_k \rangle^2 = \langle \Delta a_k^2 \rangle + 1/2. \qquad (13)
\end{aligned}$$

The preceding semiclassical variance results are ready for comparison with those obtained from the quantum theory. In particular, for direct detection we know that quantum photodetection gives[2]

$$\langle \Delta N^2 \rangle = \langle \Delta \hat{N}^2 \rangle = \mathrm{tr}(\hat{\rho}_a \Delta \hat{N}^2), \tag{14}$$

where $\Delta \hat{N} \equiv \hat{N} - \langle \hat{N} \rangle = \hat{N} - \mathrm{tr}(\hat{\rho}_a \hat{N})$. Similarly, for homodyne detection, quantum photodetection implies that

$$\langle \Delta \alpha_\theta^2 \rangle = \langle \Delta \hat{a}_\theta^2 \rangle = \mathrm{tr}(\hat{\rho}_a \Delta \hat{a}_\theta^2), \tag{15}$$

where $\Delta \hat{a}_\theta \equiv \hat{a}_\theta - \langle \hat{a}_\theta \rangle = \hat{a}_\theta - \mathrm{tr}(\hat{\rho}_a \hat{a}_\theta)$. Finally, for heterodyne detection, quantum photodetection theory gives

$$\langle \Delta \alpha_k^2 \rangle = \langle \Delta \hat{a}_k^2 \rangle + \langle \Delta \hat{a}_{I_k}^2 \rangle = \mathrm{tr}(\hat{\rho}_a \Delta \hat{a}_k^2) + 1/4, \quad \text{for } k = 1, 2, \tag{16}$$

where $\Delta \hat{a}_k \equiv \hat{a}_k - \langle \hat{a}_k \rangle = \hat{a}_k - \mathrm{tr}(\hat{\rho}_a \hat{a}_k)$, and $\Delta \hat{a}_{I_k} \equiv \hat{a}_{I_k} - \langle \hat{a}_{I_k} \rangle = \hat{a}_{I_k}$.

Our semiclassical and quantum variance results are summarized on Slide 5, where, for brevity, we have only shown the first quadrature results for homodyne and heterodyne detection. It is apparent from our variance results that semiclassical theory has lower bounds that can be broken within the quantum theory. Thus, for direct detection, because $\mathrm{var}(|a|^2) \geq 0$, we know that semiclassical theory can only explain cases in which $\langle \Delta N^2 \rangle \geq \langle N \rangle$, i.e., the photocount variance is *at least* equal to the mean photocount. Because $\langle \Delta N^2 \rangle = \langle N \rangle$ is a property of the Poisson distribution that characterizes semiclassical direct detection when a is deterministic, we say that $\langle \Delta N^2 \rangle < \langle N \rangle$ represents a sub-Poissonian distribution. A sub-Poissonian distribution for photon counting is therefore a signature of *non-classical* light, i.e., a quantum state whose quantum photodetection statistics *cannot* be matched by a semiclassical formula. A photon number state, for which $\langle \Delta N^2 \rangle = \langle \Delta \hat{N}^2 \rangle = 0$ is the most extreme example of a quantum state that gives sub-Poissonian statistics.

For homodyne detection, we see that $\langle \Delta \alpha_\theta^2 \rangle \geq 1/4$ in the semiclassical theory, because $\langle \Delta a_\theta^2 \rangle \geq 0$. Because $1/4$ represents the normalized value of the local oscillator shot noise, we say that $\langle \Delta \alpha_\theta^2 \rangle = 1/4$ is a shot-noise limited quadrature-measurement variance. If a quantum state yields $\langle \Delta \alpha_\theta^2 \rangle = \langle \Delta \hat{a}_\theta^2 \rangle < 1/4$ we say that this state exhibits a sub-shot-noise quadrature variance. A sub-shot-noise quadrature variance is therefore another signature of non-classical light. This signature is also referred to as quadrature noise squeezing, because the Heisenberg uncertainty principle requires that $\langle \Delta \hat{a}_\theta^2 \rangle \langle \Delta \hat{a}_{\theta+\pi/2}^2 \rangle \geq 1/16$. A squeezed state $|\beta; \mu, \nu\rangle$ with $\mu^* \nu e^{-2j\theta} > 0$ will have $\langle \Delta \hat{a}_\theta^2 \rangle = (|\mu| - |\nu|)^2/4 < 1/4$, and $\langle \Delta \hat{a}_\theta^2 \rangle \langle \Delta \hat{a}_{\theta+\pi/2}^2 \rangle = 1/16$, and so is an excellent example of a sub-shot-noise non-classical state. The situation for heterodyne detection is very similar. Here we see that $\langle \Delta \alpha_k^2 \rangle \geq 1/2$, for $k = 1, 2$, in the semiclassical theory, where equality in this expression—given by the normalized local oscillator shot noise variance—represents the shot noise limit for heterodyne detection. A quantum state

[2]That $\langle \hat{A} \rangle = \mathrm{tr}(\hat{\rho} \hat{A})$ gives the expectation of an operator $\hat{A}$ when the system is in the state characterized by the density operator $\hat{\rho}$ was shown in a homework problem.

whose $\langle \Delta\alpha_k^2 \rangle$ is less than $1/2$ is a sub-shot-noise signature of non-classical light. The squeezed state $|\beta; \mu, \nu\rangle$ with $\mu^*\nu > 0$ gives this sub-shot-noise signature for $k = 1$ and the squeezed state $|\beta; \mu, \nu\rangle$ with $\mu^*\nu < 0$ gives this sub-shot-noise signature for $k = 2$.

Variance signatures of non-classicality are widely used experimentally, but they are *not* definitive. Consider the pure state

$$|\psi\rangle \equiv (|0\rangle + |N\rangle)/\sqrt{2}, \tag{17}$$

where $N > 2$ and $|n\rangle$, for $n = 0$ or N, is a photon number state. Here quantum photodetection theory gives us

$$\Pr(N = n) = |\langle n|\psi\rangle|^2 = \begin{cases} 1/2, & \text{for } n = 0 \text{ or } N, \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

which has mean $\langle \hat{N} \rangle = N/2$ and variance $\langle \Delta\hat{N}^2 \rangle = N^2/4$. Thus, because $N^2/4 > N/2$ for $N > 2$, this state does *not* have a sub-Poissonian photon counting variance. It is a simple matter to verify—using the effect of the annihilation operator on a number state—that

$$\langle \hat{a} \rangle = 0 \quad \text{and} \quad \langle \hat{a}^2 \rangle = 0, \tag{19}$$

for this state. It then follows that

$$\langle \hat{a}_\theta \rangle = \left\langle \frac{\hat{a}e^{-j\theta} + \hat{a}^\dagger e^{j\theta}}{2} \right\rangle = 0 \tag{20}$$

and hence

$$\langle \Delta\hat{a}_\theta^2 \rangle = \langle \hat{a}_\theta^2 \rangle = \left\langle \left(\frac{\hat{a}e^{-j\theta} + \hat{a}^\dagger e^{j\theta}}{2} \right)^2 \right\rangle = \frac{N+1}{4} > 1/4. \tag{21}$$

Thus this state does *not* yield a sub-shot-noise variance in homodyne detection, regardless of the local oscillator's phase shift θ. Because heterodyne detection yields

$$\langle \Delta\alpha_k^2 \rangle = \langle \Delta\hat{a}_k^2 \rangle + 1/4, \quad \text{for } k = 1, 2, \tag{22}$$

we see that this state does *not* produce a sub-shot-noise variance in heterodyne detection. So *none* of its measurement variances for the basic photodetection configurations break out of the limits of semiclassical theory. Nevertheless, this state is *not* a classical state, i.e., its photodetection statistics *cannot* be fully described by the semiclassical theory. To see that this is so, we need to provide an explicit description of a classical state, i.e., a state *all* of whose quantum photodetection statistics coincide with the corresponding results from semiclassical theory.

Complete Characterization of Classical versus Non-Classical States

Suppose that the density operator specifying the state associated with the $\hat{a}$ mode can be written in the following form,

$$\hat{\rho}_a = \int d^2\alpha\, P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha|, \tag{23}$$

where $\{|\alpha\rangle\}$ are the coherent states and $P(\alpha,\alpha^*)$ is a classical probability density function for two real variables, $\alpha_1 = \text{Re}(\alpha)$ and $\alpha_2 = \text{Im}(\alpha)$, i.e.,

$$P(\alpha,\alpha^*) \geq 0 \quad \text{and} \quad \int d^2\alpha\, P(\alpha,\alpha^*) = 1. \tag{24}$$

Physically, this is saying that the mixed state $\hat{\rho}_a$ is a classically-random mixture—with pdf $P(\alpha,\alpha^*)$—of coherent states. Mathematically, we know that the resulting probability distributions for direct, homodyne, and heterodyne detection of this state are

$$\Pr(N=n) = \int d^2\alpha\, P(\alpha,\alpha^*)|\langle n|\alpha\rangle|^2 = \int d^2\alpha\, P(\alpha,\alpha^*)\frac{|\alpha|^{2n}e^{-|\alpha|^2}}{n!}, \tag{25}$$

$$p(\alpha_\theta) = \int d^2\beta\, P(\beta,\beta^*)|_\theta\langle\alpha_\theta|\beta\rangle|^2 = \int d^2\beta\, P(\beta,\beta^*)\frac{\exp[-2(\alpha_\theta-\beta_\theta)^2]}{\sqrt{\pi/2}}, \tag{26}$$

and

$$p(\alpha) = \int d^2\beta\, P(\beta,\beta^*)\frac{|\langle\alpha|\beta\rangle|^2}{\pi} = \int d^2\beta\, P(\beta,\beta^*)\frac{\exp[-|\alpha-\beta|^2]}{\pi}, \tag{27}$$

respectively, where $\{|\beta\rangle\}$ are coherent states and $\beta_\theta = \text{Re}(\beta e^{-j\theta})$. Identifying $p_a(\alpha) = P(\alpha,\alpha^*)$ as the classical pdf needed for a in the semiclassical theory shows that the quantum theory and semiclassical theories make *identical* predictions for these complete statistical characterizations of direct, homodyne, and heterodyne detection. Thus Eq. (23) defines the classical states. Because it requires $P(\alpha,\alpha^*)$ to be a pdf, the right-hand side of Eq. (23) is called a proper P-representation. States that do *not* have proper P-representations *must* have some photodetection statistics that *cannot* be properly quantified in the semiclassical theory. In this regard, we note that the state given in Eq. (17) does not have a proper P-representation, i.e.., it is non-classical even though none of its basic photodetection variances provide non-classical signatures.[3]

On the homework you showed that

$$P(\alpha,\alpha^*) = \int \frac{d^2\zeta}{\pi^2}\, \chi_N^{\rho_a}(\zeta^*,\zeta)e^{-\zeta\alpha^*+\zeta^*\alpha}, \tag{28}$$

[3]To see that this state does not have a proper P-representation, you should convince yourself that: (1) the only pure states that have proper P-representations are the coherent states; and (2) that $(|0\rangle + |N\rangle)/\sqrt{2}$ for $N \geq 1$ is *not* a coherent state.

i.e., the P-function is the inverse Fourier transform of the normally-ordered characteristic function,

$$\chi_N^{\rho_a}(\zeta^*,\zeta) \equiv \mathrm{tr}(\hat{\rho}_a e^{\zeta\hat{a}^\dagger} e^{-\zeta^*\hat{a}}). \tag{29}$$

We know, from previous work, that the anti-normally ordered characteristic function is related to the normally-ordered characteristic function as follows

$$\chi_A^{\rho_a}(\zeta^*,\zeta) \equiv \mathrm{tr}(\hat{\rho}_a e^{-\zeta^*\hat{a}} e^{\zeta\hat{a}^\dagger}) = \chi_N^{\rho_a}(\zeta^*\zeta)e^{-|\zeta|^2}. \tag{30}$$

Moreover, we know—again from the homework—that

$$\frac{\langle\alpha|\hat{\rho}_A|\alpha\rangle}{\pi} = \int \frac{\mathrm{d}^2\zeta}{\pi^2}\,\chi_A^{\rho_a}(\zeta^*,\zeta)e^{-\zeta\alpha^*+\zeta^*\alpha}. \tag{31}$$

Mathematically, this says that $\langle\alpha|\hat{\rho}_a|\alpha\rangle$ and $\chi_A^{\rho_a}(\zeta^*,\zeta)$ form a Fourier pair. However, it is much more important to note that

$$\frac{\langle\alpha|\hat{\rho}_a|\alpha\rangle}{\pi} = \frac{\mathrm{tr}(\hat{\rho}_a|\alpha\rangle\langle\alpha|)}{\pi}, \tag{32}$$

hence it is the pdf for heterodyne detection as

$$\hat{\Pi}(\alpha) = \frac{|\alpha\rangle\langle\alpha|}{\pi}, \tag{33}$$

is the $\hat{a}$ POVM. Putting it all together tells us that knowing the heterodyne detection statistics of $\hat{a}$ is sufficient to determine the anti-normally ordered characteristic function, which, in turn, is sufficient to determine the density operator $\hat{\rho}_a$ by means of the operator-valued inverse Fourier transform,

$$\hat{\rho}_a = \int \frac{\mathrm{d}^2\zeta}{\pi}\chi_A^{\rho_a}(\zeta^*,\zeta)e^{-\zeta\hat{a}^\dagger}e^{\zeta^*\hat{a}}, \tag{34}$$

which was derived on the homework. Hence, if we have an ensemble of identically prepared single-mode fields, and perform a heterodyne detection measurement on each one, we can use the data so obtained to obtain an estimate of their common density operator $\hat{\rho}_a$. This estimate will converge to $\hat{\rho}_a$ as the number of identically prepared systems grows without bound.

As a practical matter, the preceding heterodyne approach to measuring the density operator is not generally employed. Instead, quantum state tomography, based on homodyne detection is used. Once again we need an ensemble of identically prepared quantum states $\hat{\rho}_a$. We now measure the homodyne statistics $p(\alpha_\theta)$ for a wide variety of θ values. Because the classical characteristic function associated with the pdf $p(\alpha_\theta)$ is

$$M_{\alpha_\theta}(jv) = \langle e^{jv\hat{a}_\theta}\rangle = \mathrm{tr}\{\hat{\rho}_a \exp[(jve^{-j\theta}/2)\hat{a} + (jve^{j\theta}/2)\hat{a}^\dagger]\} = \chi_W^{\rho_a}(\zeta^*,\zeta)|_{\zeta=jve^{j\theta}/2}, \tag{35}$$

where $\chi_W^{\rho_a}(\zeta^*,\zeta)$ is the Wigner characteristic function, the projection-slice theorem from signal processing allows us to recover $\chi_W^{\rho_a}(\zeta^*,\zeta)$ from knowledge of the pdfs $\{\, p(\alpha_\theta) : 0 \le \theta \le \pi/2\,\}$.[4] Then, via

$$\chi_A^{\rho_a}(\zeta^*,\zeta) = \chi_W^{\rho_a}(\zeta^*,\zeta)e^{-|\zeta|^2/2}. \tag{36}$$

and Eq. (34) we get back to the density operator $\hat{\rho}_a$.

Optical Waveguide Tap

Equipped with our knowledge of single-mode photodetection, let's reexamine the optical waveguide tap that we considered in Lecture 1. The configuration of interest is shown, in its semiclassical instantiation, on Slide 7. A classical single-mode input signal with phasor $a_{s_{\rm in}}$ and a classical single-mode tap input with phasor $a_{t_{\rm in}}$ enter a lossless, passive, fused-fiber coupler with transmissivity T, where $0 < T < 1$. The resulting phasors at the signal and tap output ports are therefore

$$a_{s_{\rm out}} = \sqrt{T}\,a_{s_{\rm in}} + \sqrt{1-T}\,a_{t_{\rm in}} \quad \text{and} \quad a_{t_{\rm out}} = \sqrt{1-T}\,a_{s_{\rm in}} - \sqrt{T}\,a_{t_{\rm in}}. \tag{37}$$

We'll assume that the signal input is deterministic and that the tap input is zero. We'll consider what happens if we perform homodyne detection at the signal input port *or* we perform homodyne detection at *both* the signal and the tap output ports. In all these homodyne detectors, we will take the local oscillator to be in phase with the classical field phasor. We then get the signal-to-noise (SNR) results posited in Lecture 1 and shown on Slide 7, viz.,

$$\mathrm{SNR}_{\rm in} = 4|a_{s_{\rm in}}|^2 \tag{38}$$

$$\mathrm{SNR}_{\rm out} = 4|a_{s_{\rm out}}|^2 = 4T|a_{s_{\rm in}}|^2 \tag{39}$$

$$\mathrm{SNR}_{\rm tap} = 4|a_{t_{\rm out}}|^2 = 4(1-T)|a_{s_{\rm in}}|^2. \tag{40}$$

Ideal semiclassical homodyne detection—with the local oscillator phase matched to the signal—results in an SNR given by 4 times the classical photon flux. So, because the fused fiber coupler is both lossless and passive, we have that

$$\mathrm{SNR}_{\rm in} = \mathrm{SNR}_{\rm out} + \mathrm{SNR}_{\rm tap}, \tag{41}$$

which shows that there is a fundamental SNR tradeoff between the signal output and tap output ports. With classical electromagnetic waves and the homodyne noise being local oscillator shot noise, what else could possibly be done? In quantum theory, however, there *is* something else that could be done.

[4]There are interesting signal processing issues that arise in quantum state tomography, but we do not have the time to treat them.

Slide 8 summarizes the quantum theory for the waveguide tap. Now, the phasors from the semiclassical theory become photon annihilation operators, and the beam splitter relation becomes operator valued, viz.,[5]

$$\hat{a}_{s_{\rm out}} = \sqrt{T}\,\hat{a}_{s_{\rm in}} + \sqrt{1-T}\,\hat{a}_{t_{\rm in}} \quad \text{and} \quad \hat{a}_{t_{\rm out}} = \sqrt{1-T}\,\hat{a}_{s_{\rm in}} - \sqrt{T}\,\hat{a}_{t_{\rm in}}. \tag{42}$$

If we take the signal input to be in the coherent state $|a_{s_{\rm in}}\rangle$, the tap input to be in the vacuum state $|0\rangle$, and the local oscillators used to homodyne at either the signal input port *or* both the signal output and tap output ports to be strong coherent states that are in phase with the mean signal at these locations then the quantum theory reproduces the SNR formulas that we have just exhibited for the semiclassical theory. This should come as no surprise, because all the fields involved are coherent states and coherent states have proper P-representations, i.e., they are classical states. Despite this quantitative equivalence, there is a fundamental qualitative difference between the two derivations of the preceding SNR formulas. In the semiclassical treatment the noise is local oscillator shot noise, whereas in the quantum theory the noise is the quantum noise of the field quadrature measurement. Therefore, if we allow tap input to be in an arbitrary zero-mean-field ($\langle \hat{a}_{t_{\rm in}}\rangle = 0$) state, while leaving the signal input and the local oscillators as they were, we find that

$$\begin{aligned} {\rm SNR}_{\rm in} &= 4|a_{s_{\rm in}}|^2 && (43)\\ {\rm SNR}_{\rm out} &= \frac{4T|a_{s_{\rm in}}|^2}{T + 4(1-T)\langle \Delta\hat{a}^2_{t_\theta}\rangle} && (44)\\ {\rm SNR}_{\rm tap} &= \frac{4(1-T)|a_{s_{\rm in}}|^2}{(1-T) + 4T\langle \Delta\hat{a}^2_{t_\theta}\rangle}, && (45) \end{aligned}$$

where $\hat{a}_{t_\theta} = {\rm Re}(\hat{a}_{t_{\rm in}} e^{-j\theta})$ for θ defined by $a_{s_{\rm in}} = |a_{s_{\rm in}}|e^{j\theta}$. Now, it is clear that if we squeeze the $\hat{a}_{t_\theta}$ quadrature so that, for the given T value, we can neglect the tap-input noise contributions in the ${\rm SNR}_{\rm out}$ and ${\rm SNR}_{\rm tap}$ expressions we obtain

$${\rm SNR}_{\rm out} \approx {\rm SNR}_{\rm tap} \approx {\rm SNR}_{\rm in} = 4|a_{s_{\rm in}}|^2, \tag{46}$$

in clear violation of the SNR tradeoff that exists in the semiclassical theory.

Slide 9 is a parametric plot of ${\rm SNR}_{\rm tap}/{\rm SNR}_{\rm in}$ versus ${\rm SNR}_{\rm out}/{\rm SNR}_{\rm in}$, as T varies from 0 to 1, for the semiclassical and the quantum theories when the latter employs a squeezed-vacuum tap input state with $4\langle \Delta\hat{a}^2_{t_\theta}\rangle = 0.1$, i.e., for 10 dB of quadrature noise squeezing. We see that there is a dramatic difference between the two curves, in that the performance of the squeezed-state waveguide tap pushes well up toward the perfect tap, ${\rm SNR}_{\rm tap}/{\rm SNR}_{\rm in} = {\rm SNR}_{\rm out}/{\rm SNR}_{\rm in} = 1$, corner, whereas the semiclassical performance rides on the ${\rm SNR}_{\rm tap}/{\rm SNR}_{\rm in} = 1 - {\rm SNR}_{\rm out}/{\rm SNR}_{\rm in}$ diagonal. Yet,

[5] On the homework you will show that this quantum beam splitter relation makes sense in that it conserves photon number and commutator brackets.

despite this wonderful performance advantage, squeezed-state waveguide taps have *not* revolutionized optical networking. If you ask why, then I will answer *loss*.

We will treat the quantum mechanics of (single-mode) attenuation and (single-mode) amplification starting next time. Nevertheless, let's jump start that discussion today by specifying the quantum photodetection theory for a *slightly* imperfect photodetector, i.e., one whose quantum efficiency is *not* unity. By this I mean that the absorption of one photon does *not* necessarily result in one charge-q impulse in the photodetector's output current, but all the other detector characteristics are as we have been assuming. As shown on Slide 10, we can account for a detector's having a quantum efficiency η by the following artifice. Let $\hat{a}$ be the annihilation operator of the single-mode field that illuminates the detector's light-sensitive region during the observation interval. We *define* a new annihilation operator, $\hat{a}'$, by the relation

$$\hat{a}' = \sqrt{\eta}\,\hat{a} + \sqrt{1-\eta}\,\hat{a}_\eta, \tag{47}$$

where $0 \leq \eta \leq 1$ is the detector's quantum efficiency and $\hat{a}_\eta$ is a photon annihilation operator for a fictitious mode that is in its vacuum state. Note that $\hat{a}_\eta$ and $\hat{a}_\eta^\dagger$ both commute with $\hat{a}$ and with $\hat{a}^\dagger$. Thus our $\hat{a}'$ definition has the appearance of a beam splitter in which only a fraction, η, of the incoming signal photons are transmitted. Quantum photodetection theory for a photodetector with sub-unity quantum efficiency then states that:

- Direct detection realizes the $\hat{a}'^\dagger\hat{a}'$ measurement.
- Homodyne detection realizes the $\hat{a}'_\theta = \mathrm{Re}(\hat{a}'e^{-j\theta})$ measurement.
- Heterodyne detection realizes the $\hat{a}'$ positive operator-valued measurement.

Redoing the SNR evaluations for the squeezed-state waveguide tap when the *output* port homodyne measurements use quantum efficiency η detectors but the *input* port detction is still done at unity quantum efficiency then leads to[6]

$$\mathrm{SNR}_{\mathrm{in}} = 4|a_{s_{\mathrm{in}}}|^2 \tag{48}$$

$$\mathrm{SNR}_{\mathrm{out}} = \frac{4\eta T|a_{s_{\mathrm{in}}}|^2}{\eta T + (1-\eta) + 4\eta(1-T)\langle\Delta\hat{a}_{t_\theta}^2\rangle} \tag{49}$$

$$\mathrm{SNR}_{\mathrm{tap}} = \frac{4\eta(1-T)|a_{s_{\mathrm{in}}}|^2}{\eta(1-T) + (1-\eta) + 4\eta T\langle\Delta\hat{a}_{t_\theta}^2\rangle}, \tag{50}$$

Now if we squeeze the noise in the $\hat{a}_{t_\theta}$ quadrature to the point that it can be neglected, we do *not* obtain ideal $\mathrm{SNR}_{\mathrm{tap}}/\mathrm{SNR}_{\mathrm{in}} = \mathrm{SNR}_{\mathrm{out}}/\mathrm{SNR}_{\mathrm{in}} = 1$ behavior. Slide 11 shows

[6]Taking the $\hat{a}_{t_{\mathrm{in}}}$ mode to be in its vacuum state gives $\langle\Delta\hat{a}_{t_\theta}^2\rangle = 1/4$, and reduces these formulas to those obtained from the semiclassical theory of photodetection with quantum efficiency η.

what transpires when $\eta = 0.7$ and we either use a vacuum-state tap input or a 10 dB squeezed tap input. In the latter case we still get performance that exceeds the former, but it barely crosses the "semiclassical frontier", $\mathrm{SNR}_{\mathrm{tap}}/\mathrm{SNR}_{\mathrm{in}} = 1 - \mathrm{SNR}_{\mathrm{out}}/\mathrm{SNR}_{\mathrm{in}}$. Physically, the degradation in our waveguide tap's performance comes from the zero-point fluctuations that are introduced by the $\hat{a}_\eta$ mode.

The Road Ahead

In the next lecture we shall study the single-mode quantum theories for linear attenuation and linear amplification. In both cases we shall find that commutator preservation—i.e., ensuring that the proper Heisenberg uncertainty principle is obeyed at the output of these linear transformations—dictates fundamental differences from what we would assume in classical studies of these systems.