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Optical heterodyne detection and the  $\hat{a}$  POVM

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## Introduction

We are close to completing our development of single-mode photodetection—in both its semiclassical and quantum forms—with the principal remaining task being the treatment of optical heterodyne detection. Heterodyne detection is the physical realization of the  $\hat{a}$  positive operator-valued measurement. Moreover, its analysis will connect with the notion that POVMs that are not observables can be regarded as observables on an enlarged—signal  $\otimes$  ancilla—state space. Before turning to heterodyne detection, we shall briefly reprise what was done last time, i.e., the single-mode semiclassical and quantum theories of direct detection and balanced homodyne detection with ideal photodetectors.

## Reprise of Direct Detection

Slide 3 shows our quantum description of a single-mode field. It is a positive-frequency field operator,  $\hat{E}_z(x, y, t)$ , that has only one spatio-temporal mode which is not in its vacuum state. Here we have taken that excited mode to be a monochromatic  $+z$ -going plane-wave pulse over the detector's photosensitive region  $\mathcal{A}$  during the detection interval  $0 \leq t \leq T$ . The “other modes” must be included, for a full quantum field description, because their vacuum states carry zero-point fluctuations that could, potentially, influence the photodetection statistics. Note that the  $\hat{a}$  operator appearing in the excited  $\hat{E}_z(x, y, t)$  mode is a photon annihilation operator, i.e., it has the canonical commutator  $[\hat{a}, \hat{a}^\dagger] = 1$  with its adjoint, the photon creation operator. Later this semester, when we cover continuous-time photodetection, we will see that all the other modes on Slide 2 are also characterized by photon annihilation operators, so that the entire quantized electromagnetic field comprises an infinite collection of quantum harmonic oscillators.

The quantum theory of photodetection for the single-mode field dictates that the final-count variable,

$$N \equiv \frac{1}{q} \int_0^T du i(u), \quad (1)$$

takes on non-negative integer values,  $n = 0, 1, 2, \dots$ , with probabilities

$$\Pr(N = n \mid \text{state} = |\psi\rangle) = |\langle n | \psi \rangle|^2, \quad (2)$$

when the excited mode is in the state  $|\psi\rangle$ , where  $\{|n\rangle\}$  are the photon-number states, i.e., the eigenkets of  $\hat{N} \equiv \hat{a}^\dagger \hat{a}$ . Inasmuch as our Axiom 3 tells us that these are the statistics of the  $\hat{N}$  observable, we can say that single-mode direct detection with an ideal photodetector realizes the  $\hat{N}$  measurement. Specifically, all statistics associated with the classical outcome  $N$  equal the corresponding statistics of the observable  $\hat{N}$ . For example,  $\langle N \rangle = \langle \hat{N} \rangle$ ,  $\langle \Delta N^2 \rangle = \langle \Delta \hat{N}^2 \rangle$ , and  $\langle e^{jvN} \rangle = \langle e^{jv\hat{N}} \rangle$ , etc. To denote this equivalence of a classical random variable to measurement of a quantum operator we write  $N \leftrightarrow \hat{N}$ .

Why don't the "other modes" on Slide 3 contribute to the statistics of  $N$ ? The full description of  $\hat{E}_z(x, y, t)$  on  $(x, y) \in \mathcal{A}$  and  $0 \leq t \leq T$  is as follows:

$$\hat{E}_z(x, y, t) = \sum_k \hat{a}_k \phi_k(x, y, t), \quad (3)$$

where  $\{\phi_k(x, y, t)\}$  is a complete orthonormal set of functions on  $(x, y, t) = \mathcal{A} \times [0, T]$ , i.e.,

$$\int_{\mathcal{A}} dx dy \int_0^T dt \phi_j^*(x, y, t) \phi_k(x, y, t) = \delta_{jk}, \quad (4)$$

and

$$\sum_k \phi_k^*(x, y, t) \phi_k(x', y', t') = \delta(x - x') \delta(y - y') \delta(t - t'), \quad (5)$$

for  $(x, y), (x', y') \in \mathcal{A}$  and  $t, t' \in [0, T]$ . So, taking  $\phi_1(x, y, t) = e^{-j\omega t} / \sqrt{AT}$ , we can say that our single-mode field from Slide 3 has its  $k = 1$  mode excited with the rest of its modes being in their vacuum states. The continuous time theory of photodetection—which will see later this semester—teaches that the final count is equivalent, in the sense described in the previous paragraph, to the total photon number operator,

$$\hat{N}_T \equiv \int_{\mathcal{A}} dx dy \int_0^T dt \hat{E}_z^\dagger(x, y, t) \hat{E}_z(x, y, t) = \sum_k \hat{a}_k^\dagger \hat{a}_k, \quad (6)$$

where the last equality follows from Parseval's theorem for the (operator-valued) generalized Fourier series. Because  $\{\hat{N}_k \equiv \hat{a}_k^\dagger \hat{a}_k : k = 2, 3, \dots\}$  are all vacuum-state modes, their measurements all yield zero-valued outcomes with probability one. Hence only the excited mode  $\hat{a} e^{-j\omega t} / \sqrt{AT}$  from Slide 3 contributes to the final count variable in the direct detection setup shown on Slide 4.

Now let us see how to connect the quantum theory of single-mode direct detection to the semiclassical view of the same configuration. Today, rather than specify the semiclassical case and compare it to the quantum formulation, let us choose to put

the single excited mode of the quantum field into the coherent state  $|\alpha\rangle$  and see what transpires. In this case, (2) becomes the Poisson distribution with mean  $|\alpha|^2$ , viz.,

$$\Pr(N = n \mid \text{state} = |\alpha\rangle) = \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!}, \quad \text{for } n = 0, 1, 2, \dots \quad (7)$$

Suppose, however, that the single-mode field is *not* in a pure state  $|\alpha\rangle$ , but is in some classically random mixture of such states, i.e., there is a classical probability density function  $p(\alpha)$  such that the density operator for the  $\hat{a}$  mode is<sup>1</sup>

$$\hat{\rho} = \int d^2\alpha p(\alpha) |\alpha\rangle\langle\alpha|. \quad (8)$$

Standard results from probability theory then tell us that

$$\Pr(N = n) = \int d^2\alpha p(\alpha) \Pr(N = n \mid \text{state} = |\alpha\rangle) \quad (9)$$

$$= \int d^2\alpha p(\alpha) \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!}, \quad \text{for } n = 0, 1, 2, \dots \quad (10)$$

This result is the quantum origin of the semiclassical theory of single-mode direct detection. Say that the detector is illuminated by the single-mode *classical* field,

$$E_z(x, y, t) = \frac{ae^{-j\omega t}}{\sqrt{AT}}, \quad \text{for } (x, y) \in \mathcal{A} \text{ and } 0 \leq t \leq T, \quad (11)$$

where  $a$  is a complex-valued random variable. If we take

$$\Pr(N = n \mid a = \alpha) = \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!}, \quad \text{for } n = 0, 1, 2, \dots, \quad (12)$$

and let  $a$  have the joint probability density  $p(\alpha)$  for  $\alpha_1 = \text{Re}(\alpha)$  and  $\alpha_2 = \text{Im}(\alpha)$ , then we get a semiclassical theory for  $\{\Pr(N = n)\}$  that coincides with the quantum theory for this probability mass function for *all* classical probability density functions  $p(\alpha)$ .

So, within the regime of density operators that are classically-random mixtures of coherent states we have that the quantum and semiclassical theories of single-mode direct detection are *quantitatively* indistinguishable. However, even in this regime the two theories are *qualitatively* different, in that the quantum theory ascribes the noise in single-mode direct detection of a pure state to the quantum natures of the light beam and the operator describing the measurement that is being made, but the semiclassical theory ascribes the noise in single-mode direct detection of a deterministic field to the shot noise associated with the discreteness of the electron

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<sup>1</sup>See Problem Set 3 for more about density operators and mixed quantum states.

charge in the photodetector. The most stark contrast between the quantum and semiclassical views of single-mode direct detection then occurs when the illumination is in a non-vacuum photon number state  $|m\rangle$ . Quantum photodetection tells us that  $N = m$  will occur with probability one, but the semiclassical theory can never predict  $\langle \Delta N^2 \rangle < \langle N \rangle$ , so its Poisson distribution for deterministic illumination *cannot* account for ideal direct detection of a non-vacuum photon number state.

## Reprise of Balanced Homodyne Detection

Slides 5 through 7 review what was done last time for single-mode balanced homodyne detection. Let's take another look at this setup starting, as we just did for direct detection, from the quantum theory. A single-mode signal field and a single-mode local oscillator field are combined on a 50/50 beam splitter after which they illuminate a pair of ideal photodetectors. From our direct detection results, we know that  $N_{\pm} \leftrightarrow \hat{N}_{\pm}$ , i.e., the final counts from these two detectors are equivalent to the photon number-operator measurements<sup>2</sup>

$$\hat{N}_{\pm} \equiv \hat{a}_{\pm}^{\dagger} \hat{a}_{\pm}, \quad \text{where} \quad \hat{a}_{\pm} \equiv \frac{\hat{a}_S \pm \hat{a}_{LO}}{\sqrt{2}}. \quad (13)$$

It follows that the classical random variable output,  $\alpha_{\theta}$ , from the balanced homodyne setup obeys  $\alpha_{\theta} \leftrightarrow \lim_{N_{LO} \rightarrow \infty} (\hat{N}_{+} - \hat{N}_{-}) / 2\sqrt{N_{LO}}$ , which ultimately becomes  $\alpha_{\theta} \leftrightarrow \text{Re}(\hat{a}_S e^{-j\theta})$ , i.e., the  $\theta$ -quadrature of the signal field's annihilation operator  $\hat{a}_S$ . Thus, if

$$\hat{a}_{\theta} |\alpha_{\theta}\rangle_{\theta} = \alpha_{\theta} |\alpha_{\theta}\rangle_{\theta}, \quad \text{for } -\infty < \alpha_{\theta} < \infty \quad (14)$$

defines the complete orthonormal (in the delta-function sense) eigenkets of  $\hat{a}_{\theta} \equiv \text{Re}(\hat{a}_S e^{-j\theta})$ , then we know (from Axiom 3a) that the outcome of single-mode balanced homodyne detection will be a continuous random variable  $\alpha_{\theta}$  with probability density function

$$p(\alpha_{\theta} | \text{state} = |\psi\rangle) = |\langle \alpha_{\theta} | \psi \rangle|^2, \quad \text{for } -\infty < \alpha_{\theta} < \infty, \quad (15)$$

when the signal mode is in the state  $|\psi\rangle$ .

Once again we connect the quantum theory to the semiclassical treatment by assuming that the excited signal mode is in the coherent state  $|\beta\rangle$ . In this case we get<sup>3</sup>

$$p(\alpha_{\theta} | \text{state} = |\beta\rangle) = \frac{e^{-2(\alpha_{\theta} - \beta_{\theta})^2}}{\sqrt{\pi/2}}, \quad \text{where } \beta_{\theta} \equiv \text{Re}(\beta e^{-j\theta}), \quad (16)$$

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<sup>2</sup>The discussion of "other modes" from our direct detection treatment shows why only a single mode of the signal and a single mode of the local oscillator contribute to the statistics of these final counts.

<sup>3</sup>This result follows from our work in Lecture 7 on quantum characteristic functions. In particular, we have that  $M_{a_{\theta}}(jv) \equiv \langle e^{jv\hat{a}_{\theta}} \rangle = \chi_W(\zeta^*, \zeta)|_{\zeta=jve^{j\theta}/2} = e^{jv\beta_{\theta} - v^2/8}$ , where the last equality uses the fact that the signal field is in the coherent state  $|\beta\rangle$ .

i.e.,  $\alpha_\theta$  is a variance-1/4 Gaussian random variable whose mean value is the  $\theta$ -quadrature of the coherent state's eigenvalue. Now, if the single-mode field is in a classical mixture of coherent states specified by the density operator,

$$\hat{\rho} = \int d^2\beta p(\beta) |\beta\rangle\langle\beta|, \quad (17)$$

where  $p(\beta)$  is a classical probability density for a pair of real-valued random variables,  $\beta_1 = \text{Re}(\beta)$  and  $\beta_2 = \text{Im}(\beta)$ , we find that the *unconditional* statistics for the balanced homodyne detector's output are given by the probability density function

$$p(\alpha_\theta) = \int d^2\beta p(\beta) \frac{e^{-2(\alpha_\theta - \beta_\theta)^2}}{\sqrt{\pi/2}}, \quad \text{for } -\infty < \alpha_\theta < \infty. \quad (18)$$

This result is the quantum origin of the semiclassical theory of single-mode balanced homodyne detection. Say that the 50/50 beam splitter is illuminated by single-mode *classical* fields,

$$E_S(x, y, t) = \frac{a_S e^{-j\omega t}}{\sqrt{AT}} \quad \text{and} \quad E_{LO}(x, y, t) = \frac{a_{LO} e^{-j\omega t}}{\sqrt{AT}}, \quad (19)$$

where  $a_S$  is a complex-valued random variable and  $a_{LO} = \sqrt{N_{LO}} e^{j\theta}$  with  $N_{LO} \rightarrow \infty$ . Starting from the Poisson distributions

$$\text{Pr}(N_\pm = n_\pm \mid a_\pm = \alpha_\pm) = \frac{|\alpha_\pm|^{2n_\pm} e^{-|\alpha_\pm|^2}}{n_\pm!}, \quad \text{for } n_\pm = 0, 1, 2, \dots, \quad (20)$$

and the statistical independence of the shot noises of physically separate detectors, we can assign the joint probability density  $p(\alpha)$  to  $\alpha_1 = \text{Re}(\alpha)$  and  $\alpha_2 = \text{Im}(\alpha)$ , and use the Central Limit Theorem to get a semiclassical theory for  $p(\alpha_\theta)$  that coincides with the quantum theory for this probability density function for *all* classical probability density functions  $p(\alpha)$  and all phase shifts  $\theta$ .

Thus, within the regime of density operators that are classically-random mixtures of coherent states we have that the quantum and semiclassical theories of single-mode balanced homodyne detection are *quantitatively* indistinguishable. However, even in this regime the two theories are *qualitatively* different, in that the quantum theory ascribes the noise in single-mode homodyne detection of a pure state to the quantum natures of the light beam and the operator describing the measurement that is being made, but the semiclassical theory ascribes the noise in single-mode homodyne detection of a deterministic field to the shot noise of the strong local oscillator. As a result, if we illuminate the balanced homodyne detector with a squeezed-state single-mode signal field,  $|\beta; \mu, \nu\rangle$  with  $\mu, \nu > 0$ , and measure the  $\hat{a}_{S_1} = \hat{a}_{\theta=0}$  quadrature, then the quantum theory leads to a measurement variance of  $(\mu - \nu)^2/4 < 1/4$  but semiclassical photodetection can never predict  $\langle \Delta\alpha_\theta^2 \rangle < 1/4$ .

## Balanced Heterodyne Detection

We are now ready for the last of the three basic configurations for single-mode photodetection: balanced heterodyne detection, as shown on Slide 8. Here, a single-mode signal field of frequency  $\omega$  is combined, on a 50/50 beam splitter, with a strong local oscillator field of frequency  $\omega - \omega_{\text{IF}}$ , where  $\omega_{\text{IF}}$  is an intermediate frequency (IF), i.e., a radio or microwave frequency low enough to be handled by the post-photodetection electronics. Because photodetectors are essentially square-law devices, we know that the photocurrents  $i_+(t)$  and  $i_-(t)$  will contain signal  $\times$  local oscillator beats at frequency  $\omega_{\text{IF}}$ . A frequency- $\omega_{\text{IF}}$  waveform,  $x(t)$ , on the time interval  $0 \leq t \leq T$  can be written in quadrature form as

$$x(t) = x_c \cos(\omega_{\text{IF}}t) + x_s \sin(\omega_{\text{IF}}t), \quad (21)$$

where

$$x_c = \frac{2}{T} \int_0^T dt x(t) \cos(\omega_{\text{IF}}t) \quad \text{and} \quad x_s = \frac{2}{T} \int_0^T dt x(t) \sin(\omega_{\text{IF}}t). \quad (22)$$

Thus the post-photodetection processing shown in Slide 8 extracts these quadrature coefficients from the normalized photocurrent difference,  $[i_+(t) - i_-(t)]/q\sqrt{N_{\text{LO}}}$ .

For the semiclassical treatment of balanced heterodyne detection, we take the signal and local oscillator fields to be

$$E_S(x, y, t) = \frac{a_S e^{-j\omega t}}{\sqrt{AT}} \quad \text{and} \quad E_{\text{LO}}(x, y, t) = \frac{a_{\text{LO}} e^{-j(\omega - \omega_{\text{IF}})t}}{\sqrt{AT}}, \quad (23)$$

where  $a_S$  is a (deterministic) complex number and  $a_{\text{LO}} = \sqrt{N_{\text{LO}}}$  with  $N_{\text{LO}} \rightarrow \infty$ . Then, we can find the mean photocurrents from

$$\langle i_{\pm}(t) \rangle = q \int_{\mathcal{A}} dx dy |E_{\pm}(x, y, t)|^2 = \frac{q}{2T} |a_S e^{-j\omega t} \pm a_{\text{LO}} e^{-j(\omega - \omega_{\text{IF}})t}|^2 \quad (24)$$

$$= \frac{q}{2T} [|a_S|^2 + |a_{\text{LO}}|^2 \pm 2\text{Re}(a_S a_{\text{LO}}^* e^{-j\omega_{\text{IF}}t})], \quad (25)$$

where we have used the continuous-time theory of semiclassical photodetection to obtain the first equality. It is now easy to verify that

$$\lim_{N_{\text{LO}} \rightarrow \infty} \langle [i_+(t) - i_-(t)] \rangle / q\sqrt{N_{\text{LO}}} = \frac{2\text{Re}(a_S e^{-j\omega_{\text{IF}}t})}{T}, \quad (26)$$

from which we get

$$\langle \alpha_k \rangle = a_{S_k}, \quad \text{for } k = 1, 2, \quad (27)$$

where  $a_{S_1} = \text{Re}(a_S)$  and  $a_{S_2} = \text{Im}(a_S)$  are the quadrature components of  $a_S$ .

To complete our derivation of the semiclassical statistics of single-mode balanced heterodyne detection, we need to find the behavior of the noises  $\Delta\alpha_k \equiv \alpha_k - \langle \alpha_k \rangle$ , for

$k = 1, 2$ . A proper derivation of the noise behavior requires random process theory that is not assumed in our prerequisites and which we will not develop now. So, we will get by with a little handwaving. It should be clear that as we make the local oscillator's strength grow without bound its shot noise will dominate the fluctuation behavior of the  $\{\alpha_k\}$ . Moreover, because the high mean-value limit of a Poisson random variable is Gaussian,<sup>4</sup> you should not be surprised when you are told that, in semiclassical theory, the  $\{\alpha_k\}$  for single-mode balanced heterodyne detection are statistically independent variance-1/2 Gaussian random variables whose mean values are given by (27).

Before turning to the quantum treatment of balanced heterodyne detection, there is one additional point worth making. The preceding development presumed that  $a_S$  was deterministic, i.e., a known complex number. What happens if  $a_S$  is a complex-valued random variable with probability density function  $p(\alpha_S)$  for its real and imaginary parts to be  $\alpha_{S1} = \text{Re}(\alpha_S)$  and  $\alpha_2 = \text{Im}(\alpha_S)$ , respectively? The answer is obvious. The known  $a_S$  results become the *conditional* statistics of the balanced heterodyne measurement, i.e.,

$$p(\alpha \mid a_S = \alpha_S) = \frac{e^{-|\alpha - \alpha_S|^2}}{\pi}, \quad (28)$$

is the conditional probability density for the balanced heterodyne detection system's output to be  $\alpha$  given that  $a_S = \alpha_S$ . The unconditional statistics are found by averaging over the probability density for  $a_S$ ,

$$p(\alpha) = \int d^2\alpha_S p(\alpha_S) \frac{e^{-|\alpha - \alpha_S|^2}}{\pi}. \quad (29)$$

Slide 10 shows the field operators that we will need to obtain the quantum theory of single-mode balanced heterodyne detection. The local oscillator field operator is the obvious generalization of what we used for balanced homodyne detection. All that has changed in going from the homodyne case to the heterodyne case is that the frequency of the strong, coherent-state local oscillator mode has been shifted to be offset by the intermediate frequency from the frequency of the excited signal mode.<sup>5</sup> We've done something more subtle, on Slide 10, in spelling out the field operator for the signal field. Here, in addition to the excited signal mode—governed by annihilation operator  $\hat{a}_S$ —we have explicitly called out a particular *unexcited* mode that has the same  $+z$ -going plane wave spatial characteristic as the excited signal and local oscillator modes, but is at a frequency that is  $\omega_{\text{IF}}$  *below* that of the local oscillator, whereas the signal mode's frequency is  $\omega_{\text{IF}}$  *above* the LO frequency. This

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<sup>4</sup>A simple example of this asymptotic behavior is as follows. Suppose that  $N$  is a Poisson random variable with mean  $m$ , then we have that  $z \equiv (N - m)/\sqrt{m}$  is a zero-mean, variance-1 random variable. Moreover, the characteristic function of  $z$  is  $M_z(jv) = M_N(jv/\sqrt{m})e^{-jv\sqrt{m}} = e^{m(e^{jv/\sqrt{m}} - 1)}e^{-jv\sqrt{m}}$ . In the limit  $m \rightarrow \infty$ , this gives  $M_z(jv) \rightarrow e^{-v^2/2}$ , which is the characteristic function of a zero-mean, variance-1 Gaussian random variable.

<sup>5</sup>We have also set the phase of the local oscillator's coherent state eigenvalue equal to zero.



unexcited mode is governed by the annihilation operator  $\hat{a}_I$ , where  $I$  stands for *image* field. Physically, the square-law nature of photodetection—which prevails in *both* the semiclassical and quantum descriptions—implies that both the signal and image fields beat with the local oscillator to produce terms at the intermediate frequency  $\omega_{\text{IF}}$ . In semiclassical theory an unexcited image field has value zero, and so contributes nothing to the balanced heterodyne detection system’s output. In quantum theory, however, an unexcited image field carries zero-point fluctuations that contribute noise to the balanced heterodyne detection system’s output.

Paralleling what we did for the semiclassical version of balanced heterodyne detection we use a result from continuous-time quantum photodetection to say that  $i_{\pm}(t) \leftrightarrow \hat{i}_{\pm}(t)$ , where<sup>6</sup>

$$\hat{i}_{\pm}(t) = q \int_{\mathcal{A}} dx dy \hat{E}_{\pm}^{\dagger}(x, y, t) \hat{E}_{\pm}(x, y, t) \quad (30)$$

$$\begin{aligned} &= \frac{q}{2T} (\hat{a}_S e^{-j\omega t} + \hat{a}_I e^{-j(\omega-2\omega_{\text{IF}})t} \pm \hat{a}_{\text{LO}} e^{-j(\omega-\omega_{\text{IF}})t})^{\dagger} \\ &\times (\hat{a}_S e^{-j\omega t} + \hat{a}_I e^{-j(\omega-2\omega_{\text{IF}})t} \pm \hat{a}_{\text{LO}} e^{-j(\omega-\omega_{\text{IF}})t}) \end{aligned} \quad (31)$$

$$\begin{aligned} &= \frac{q}{2T} [\hat{a}_S^{\dagger} \hat{a}_S + \hat{a}_I^{\dagger} \hat{a}_I + \hat{a}_{\text{LO}}^{\dagger} \hat{a}_{\text{LO}} + 2\text{Re}(\hat{a}_S \hat{a}_I^{\dagger} e^{-j2\omega_{\text{IF}}t}) \pm 2\text{Re}(\hat{a}_S \hat{a}_{\text{LO}}^{\dagger} e^{-j\omega_{\text{IF}}t}) \\ &\pm 2\text{Re}(\hat{a}_I^{\dagger} \hat{a}_{\text{LO}} e^{-j\omega_{\text{IF}}t})]. \end{aligned} \quad (32)$$

Next, we use this result to evaluate the operator equivalents of the balanced heterodyne outputs, i.e.,  $\hat{\alpha}_k \leftrightarrow \alpha_k$ , for  $k = 1, 2$ , and find

$$\hat{\alpha} \equiv \hat{\alpha}_1 + j\hat{\alpha}_2 = \lim_{N_{\text{LO}} \rightarrow \infty} \frac{1}{q\sqrt{N_{\text{LO}}}} \int_0^T dt [\hat{i}_+(t) - \hat{i}_-(t)] e^{j\omega_{\text{IF}}t} \quad (33)$$

$$= \lim_{N_{\text{LO}} \rightarrow \infty} \frac{\hat{a}_S \hat{a}_{\text{LO}}^{\dagger} + \hat{a}_I^{\dagger} \hat{a}_{\text{LO}}}{\sqrt{N_{\text{LO}}}} = \hat{a}_S + \hat{a}_I^{\dagger}. \quad (34)$$

Strictly speaking, our notation here has been a little cavalier. The signal, image, and local oscillator modes all have annihilation operators which reside on different Hilbert spaces,  $\mathcal{H}_S$ ,  $\mathcal{H}_I$ , and  $\mathcal{H}_{\text{LO}}$ , respectively. Thus our final result for the quantum theory of single-mode balanced heterodyne detection should really be written

$$\hat{\alpha} = \hat{a}_S \otimes \hat{I}_I + \hat{I}_S \otimes \hat{a}_I^{\dagger}, \quad (35)$$

where  $\hat{I}_S$  and  $\hat{I}_I$  are the identity operators on the signal mode and image mode Hilbert spaces. However, we will continue to use the shorter notation in what follows, with the implicit understanding that it is this tensor product form that is being considered.

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<sup>6</sup>In the second equality we have anticipated the effect of the post-photodetection quadrature-component extraction, which will imply that only the  $\hat{a}_S$ ,  $\hat{a}_I$  and  $\hat{a}_{\text{LO}}$  modes will contribute to the output of the balanced heterodyne detection system.

Equation (35) is *exactly* the commuting observables on a larger Hilbert space form of the  $\hat{a}_S$  POVM that we exhibited in the last lecture, only now we have a physical realization of the measurement and thus a physical locus—the image mode—for the ancilla that injects the extra noise into the two quadrature observations. Ordinarily, however, we do not bother with explicit recognition of the image mode, and we just say that the complex-valued classical random variable that is obtained from single-mode balanced heterodyne detection realizes the  $\hat{a}_S$  POVM. This means that all statistics of this complex-valued classical random variable  $\alpha$  coincide with the corresponding statistics of the  $\hat{a}$  operator, so we write  $\alpha \leftrightarrow \hat{a}_S$ .

It is instructive to consider the case in which the signal mode is in the coherent state  $|\beta\rangle$ . Here we find that the classical probability density for the balanced heterodyne detector's outcome is

$$p(\alpha \mid \text{state} = |\beta\rangle) = \frac{|\langle\alpha|\beta\rangle|^2}{\pi} = \frac{e^{-|\alpha-\beta|^2}}{\pi}. \quad (36)$$

In words this means that  $\alpha_1$  and  $\alpha_2$  are statistically independent, variance-1/2 Gaussian random variables with mean values  $\beta_1$  and  $\beta_2$ , exactly as found from the semiclassical theory. However, the quantum theory ascribes the noise in balanced heterodyne detection to the quantum noise on the signal and image modes, whereas the semiclassical theory says that the noise is local oscillator shot noise. When the signal mode is in a classically-random mixture of coherent states, its predictions are still in perfect quantitative agreement with those of the semiclassical theory. But, because of the difference in their interpretations of where the noise comes from, there are quantum states whose heterodyne statistics *cannot* be accounted for in the semiclassical theory. An example is the squeezed state  $|\beta; \mu, \nu\rangle$  with  $\mu, \nu > 0$ . In this case the quantum theory gives  $\langle\Delta\hat{\alpha}_1^2\rangle = [(\mu - \nu)^2 + 1]/4 < 1/2$ , whereas the semiclassical theory can never have  $\langle\Delta\alpha_1^2\rangle < 1/2$ .

## The Road Ahead

In the next lecture we shall wrap up our discussion of single-mode photodetection by compiling the non-classical signatures of states that are not coherent states or classically-random mixtures thereof. We then return to the waveguide tap that we discussed in Lecture 1 and show exactly how it works.