6.453 Quantum Optical Communication
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6.453 *Quantum Optical Communication* Lecture 4

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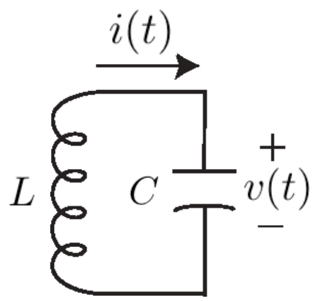
6.453 *Quantum Optical Communication* — Lecture 4

- Handouts
  - Lecture notes, slides
- Quantum Harmonic Oscillator
  - Quantization of a classical *LC* circuit
  - Annihilation and creation operators
  - Energy eigenstates — number-state kets

Classical *LC* Circuit: Undriven, Lossless Oscillation

- State Variables



capacitor charge: $q(t) = Cv(t)$

inductor flux: $p(t) = Li(t)$

- Stored Energy (Hamiltonian)

$$H = \frac{q^2(t)}{2C} + \frac{p^2(t)}{2L}$$

- Hamilton's Equations

$$\frac{\partial H(q,p)}{\partial q} = \frac{q(t)}{C} = -\dot{p}(t)$$

$$\frac{\partial H(q,p)}{\partial p} = \frac{p(t)}{L} = \dot{q}(t)$$

Classical *LC* Circuit: Undriven, Lossless Oscillation

- Nonzero Initial Conditions: $q(0), p(0)$
- Oscillation Frequency: $\omega = 1/\sqrt{LC}$
- Solutions for $t \geq 0$:

$$\mathbf{q} \equiv q(0) + jp(0)/\omega L$$

$$q(t) = \mathrm{Re}[\mathbf{q}e^{-j\omega t}]$$

$$p(t) = \mathrm{Im}[\omega L\mathbf{q}e^{-j\omega t}]$$

$$H = \frac{|\mathbf{q}|^2}{2C} = \text{constant}$$

Classical *LC* Circuit: Dimensionless Reformulation

- Assume $L = 1$
- Define Complex Envelope (Phasor):

$$a(t) \equiv a_1(t) + ja_2(t) = \sqrt{\frac{\omega}{2\hbar}}q(t) + j\sqrt{\frac{1}{2\hbar\omega}}p(t)$$

- Simple Harmonic Motion at Constant Energy

$$a(t) = ae^{-j\omega t}$$

$$H = \hbar\omega[a_1^2(t) + a_2^2(t)] = \hbar\omega|a|^2$$

Quantum *LC* Circuit: Quantum Harmonic Oscillator

- Postulate: $H, q(t), p(t)$ Become Observables $\hat{H}, \hat{q}(t), \hat{p}(t)$
- Canonical Commutation Relation: $[\hat{q}(t), \hat{p}(t)] = j\hbar$
- Dimensionless Reformulation:

$$\hat{a}(t) \equiv \hat{a}_1(t) + j\hat{a}_2(t) = \sqrt{\frac{\omega}{2\hbar}}\hat{q}(t) + j\sqrt{\frac{1}{2\hbar\omega}}\hat{p}(t)$$

$$\hat{a}(t) = \hat{a}e^{-j\omega t}$$

$$\hat{H} = \hbar\omega[\hat{a}_1^2(t) + \hat{a}_2^2(t)]$$

Quantum Harmonic Oscillator: Commutators

- Dimensionless Reformulation of Canonical Commutators:

$$[\hat{q}(t), \hat{p}(t)] = j\hbar \longrightarrow [\hat{a}_1(t), \hat{a}_2(t)] = j/2 \longleftrightarrow [\hat{a}(t), \hat{a}^\dagger(t)] = 1$$

- Hamiltonian:

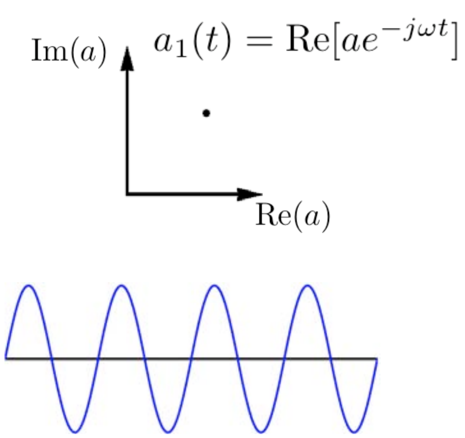
$$\hat{H} = \hbar\omega[\hat{a}_1^2(t) + \hat{a}_2^2(t)] = \hbar\omega[\hat{a}^\dagger\hat{a} + 1/2]$$

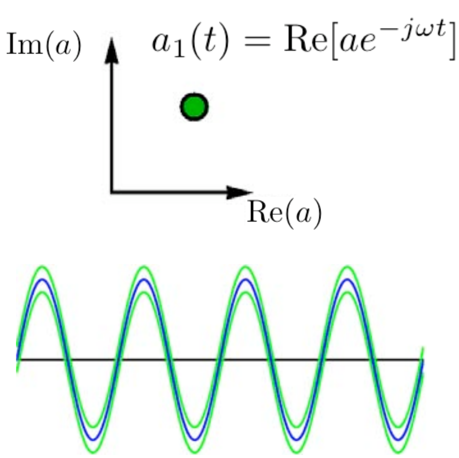
- Heisenberg Uncertainty Principle:

$$\langle\Delta\hat{a}_1^2(t)\rangle\langle\Delta\hat{a}_2^2(t)\rangle \geq \frac{1}{16}$$

Classical versus Quantum Behavior

- Classical Oscillator: Noiseless
- Quantum Oscillator: Noisy





Energy Eigenvalues and Eigenkets

- Notation:

$$\hat{H}|E_n\rangle = E_n|E_n\rangle, \quad \text{for } n = 0, 1, 2, \ldots$$

- Annihilation and Creation Operations
  - $\hat{a}|E_n\rangle$ is energy eigenket with eigenvalue $E_n - \hbar\omega$
  - $\hat{a}^\dagger|E_n\rangle$ is energy eigenket with eigenvalue $E_n + \hbar\omega$

- Minimum Energy State: $|E_0\rangle$ such that $\hat{a}|E_0\rangle = 0$

- Energy Eigenvalues: $E_n = \hbar\omega(n + 1/2)$

Number Operator and Number States

- Oscillator Energy is Quantized in $\hbar\omega$ Increments

- (Photon) Number Operator:

$$\hat{N} \equiv \hat{a}^\dagger\hat{a} = \sum_{n=0}^{\infty} n|n\rangle\langle n|$$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \text{ for } n = 1, 2, 3, \ldots$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \text{ for } n = 0, 1, 2, \ldots$$

$$\hat{H}|n\rangle = \hbar\omega(\hat{N} + 1/2)|n\rangle = \hbar\omega(n + 1/2)|n\rangle$$

Coming Attractions: Lectures 5 and 6

- Lecture 5:
  Quantum Harmonic Oscillator
  - Number measurements versus quadrature measurements
  - Coherent states and their measurement statistics

- Lecture 6:
  Quantum Harmonic Oscillator
  - Minimum uncertainty-product states
  - Squeezed states and their measurement statistics