6.453 Quantum Optical Communication
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6.453 *Quantum Optical Communication* Lecture 3

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- Announcements
  - Turn in problem set 1
  - Pick up problem set 1 solution, problem set 2, lecture notes, slides
- Fundamentals of Dirac-Notation Quantum Mechanics
  - Definitions and axioms — reprise
  - Quantum measurements — statistics
  - Schrödinger picture versus Heisenberg picture
  - Heisenberg uncertainty principle

Quantum Systems and Quantum States

- *Definition 1*:

  A quantum-mechanical system $\mathcal{S}$ is a physical system governed by the laws of quantum mechanics.

- *Definition 2*:

  The state of a quantum mechanical system at a particular time t is the sum total of all information that can be known about the system at time t. It is a ket vector $|\psi(t)\rangle$ in an appropriate Hilbert space $\mathcal{H}_{\mathcal{S}}$ of possible states. Finite energy states have unit length ket vectors, i.e., $\langle\psi(t)|\psi(t)\rangle = 1$.

Time Evolution via the Schrödinger Equation

- *Axiom 1:*

  For $t \geq 0$, an isolated system with initial state $|\psi(0)\rangle$ will reach state

$$|\psi(t)\rangle = \hat{U}(t,0)|\psi(0)\rangle$$

  where $\hat{U}(t,0)$ is the unitary time-evolution operator for the system $\mathcal{S}$. $\hat{U}(t,0)$ is obtained by solving

$$j\hbar\frac{\mathrm{d}\hat{U}(t,0)}{\mathrm{d}t} = \hat{H}\hat{U}(t,0), \quad \text{for } t \geq 0, \text{ with } \hat{U}(0,0) = \hat{I}$$

  where $\hat{H}$ is the Hamiltonian (energy) operator for $\mathcal{S}$. Equivalently, we have the Schrödinger equation

$$j\hbar\frac{\mathrm{d}|\psi(t)\rangle}{\mathrm{d}t} = \hat{H}|\psi(t)\rangle, \quad \text{for } t \geq 0, \text{ with } |\psi(0)\rangle \text{ initial condition}$$

Quantum Measurements: Observables

- *Axiom 2:*
  An observable is a measurable dynamical variable of the quantum system $\mathcal{S}$. It is represented by an Hermitian operator which has a complete set of eigenkets.

- *Axiom 3:*
  For a quantum system $\mathcal{S}$ that is in state $|\psi(t)\rangle$ at time t, measurement of the observable

$$\hat{O} \equiv \sum_n o_n |o_n\rangle\langle o_n|$$

  yields an outcome that is one of the eigenvalues, $\{o_n\}$, with

$$\Pr(\text{outcome } = o_n) = |\langle o_n|\psi(t)\rangle|^2$$

Quantum Measurements: Observables

- *Projection postulate:*
  Immediately after a measurement of an observable $\hat{O}$, with distinct eigenvalues, yields outcome o_n the state of the system becomes $|o_n\rangle$.

- *Axiom 3a:*
  For a quantum system $\mathcal{S}$ that is in state $|\psi(t)\rangle$ at time t, measurement of the observable

$$\hat{O} = \int_{-\infty}^{\infty} \mathrm{d}o\, o|o\rangle\langle o|$$

  yields an outcome that is one of the eigenvalues, o, with

$$p(o) = |\langle o|\psi(t)\rangle|^2$$

Quantum Measurements: Statistics

- Average Value of an Observable Measurement
  - Discrete eigenvalue spectrum

$$\langle \hat{O} \rangle \equiv \sum_n o_n \, \mathrm{Pr}(\mathrm{outcome} = o_n) = \sum_n o_n |\langle o_n | \psi(t) \rangle|^2 = \langle \psi(t) | \hat{O} | \psi(t) \rangle$$

  - Continuous eigenvalue spectrum

$$\langle \hat{O} \rangle \equiv \int_{-\infty}^{\infty} \mathrm{d}o \, o p(o) = \int_{-\infty}^{\infty} \mathrm{d}o \, |\langle o | \psi(t) \rangle|^2 = \langle \psi(t) | \hat{O} | \psi(t) \rangle$$

- Variance of an Observable Measurement

$$\langle \Delta \hat{O}^2 \rangle = \langle \psi(t) | \Delta \hat{O}^2 | \psi(t) \rangle, \quad \text{for } \Delta \hat{O} \equiv \hat{O} - \langle \hat{O} \rangle$$

Schrödinger versus Heisenberg Pictures

- Schrödinger Picture
  - Observables are time-independent operators
  - Between measurements, states evolve according to the Schrödinger equation

$$\{ \, |\psi(t)\rangle_S, \hat{O}_S, \hat{H}_S : t \geq 0 \, \}$$

- Heisenberg Picture
  - Between measurements, states are constant
  - Observables evolve in time according to appropriate equations of motion

$$\{ \, |\psi\rangle_H, \hat{O}_H(t), \hat{H}_H(t) : t \geq 0 \, \}$$

Converting Between Pictures

- Statistics of an Observable Measurement
  - Schrödinger picture

$$\Pr(\text{outcome} = o_n) = |{}_S\langle o_n|\psi(t)\rangle_S|^2$$

  - Heisenberg picture

$$\Pr(\text{outcome} = o_n) = |{}_H\langle o_n(t)|\psi\rangle_H|^2$$

- Invariance of Statistics to Choice of Picture

$$|o_n(t)\rangle_H = \hat{U}^\dagger(t,0)|o_n\rangle_S$$

Heisenberg Equations of Motion

- Transforming an Observable between Pictures

$$\hat{O}_H(t) = \hat{U}^\dagger(t,0)\hat{O}_S\hat{U}(t,0)$$

- Equation of Motion for $\hat{O}_H(t)$

$$j\hbar\frac{\mathrm{d}\hat{O}_H(t)}{\mathrm{d}t} = \left[\hat{O}_H(t), \hat{H}\right], \quad \text{for } t \geq 0, \text{ with } \hat{O}_H(0) = \hat{O}_S$$

- Commutator Brackets

$$\left[\hat{O}_H(t), \hat{H}\right] \equiv \hat{O}_H(t)\hat{H} - \hat{H}\hat{O}_H(t)$$

Heisenberg Uncertainty Principle

- $\hat{A}(t)$ and $\hat{B}(t)$ Noncommuting Observables

$$\left[\hat{A}(t), \hat{B}(t)\right] = j\hat{C}(t), \quad \hat{C}(t) \text{ is Hermitian}$$

- Lower Limit on Product of Individual Measurement Variances

$$\langle \Delta\hat{A}(t)^2\rangle\langle\Delta\hat{B}(t)^2\rangle \geq \frac{1}{4}|\langle\hat{C}(t)\rangle|^2$$

Coming Attractions: Lectures 4 and 5

- Lecture 4:
  Quantum Harmonic Oscillator
  - Quantization of a classical *LC* circuit
  - Annihilation and creation operators
  - Energy eigenstates — number-state kets

- Lecture 5:
  Quantum Harmonic Oscillator
  - Quadrature measurements
  - Coherent states