6.443J / 8.371J / 18.409 / MAS.865

Quantum Information Science II

Pre-reqs: 2.111 / 18.435J

Knowledge of
- quantum mechanics
- qgates / states / circuits
- Shor, Grover, QFT algorithms
- error correction
- information concepts

Not discussed: Implementations (8.422)
Complexity classes (Shor)

Projects
- Not a survey or review
- Define a problem, steps towards sol^n
- PR-like article & presentation

Lecture 1: Quantum Operation Theory

1. Density matrices
2. System-environment
3. Quantum operations
4. Operator-sum representation

1. Density Matrices

Pure states $|\psi\rangle = a|0\rangle + b|1\rangle$

Transformations are unitary: $\mathcal{U}|\psi\rangle = a'|0\rangle + b'|1\rangle$

$|\psi_{AB}\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$

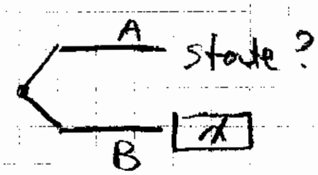
If... Want to forget a part (e.g. $|00\rangle$ term)

Cannot know a part (due to adversary, noise, etc)

Need abstraction

Then... use a density matrix!

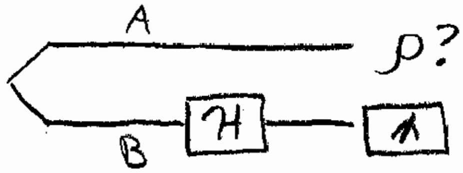
<u>Example:</u> $|\psi_{AB}\rangle = \sqrt{3/4}\,|00\rangle + \sqrt{1/4}\,|11\rangle$



A's is a statistical mixture

$\sqrt{3/4}\,|0\rangle \oplus \sqrt{1/4}\,|1\rangle$ = Mix 1

($|0\rangle$ w/prob 3/4, $|1\rangle$ w/prob 1/4)



$$\tfrac{1}{\sqrt{2}}\left[\sqrt{3/4}\,|0\rangle\left(|0\rangle + |1\rangle\right) + \sqrt{1/4}\,|1\rangle\left(|0\rangle - |1\rangle\right)\right]$$

$$\sqrt{3/8}\,|0\rangle + \sqrt{1/8}\,|1\rangle \oplus \sqrt{3/8}\,|0\rangle - \sqrt{1/8}\,|1\rangle = \text{Mix } 2$$

<u>But</u>: No experiment that Alice can perform to determine which basis Bob measured in

- The 2 representations' difference is disingenuous

A <u>density matrix</u> is a mathematical tool used to track statistical mixtures

$$|\psi_1\rangle \oplus |\psi_2\rangle \oplus \dots \oplus |\psi_n\rangle \Rightarrow \rho = \sum_k |\psi_k\rangle\langle\psi_k|$$

Recall: $|0\rangle := \begin{pmatrix}1\\0\end{pmatrix}$ $|1\rangle := \begin{pmatrix}0\\1\end{pmatrix}$

$|0\rangle\langle 0| = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$ $|0\rangle\langle 1| = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$

$|1\rangle\langle 0| = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}$ $|1\rangle\langle 1| = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$

So $\rho_{Mix1} = \frac{3}{4}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \frac{1}{4}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}3 & 0\\0 & 1\end{pmatrix}$

$\rho_{Mix2} = \frac{1}{8}\begin{pmatrix}3 & \sqrt{3}\\\sqrt{3} & 1\end{pmatrix} + \frac{1}{8}\begin{pmatrix}3 & -\sqrt{3}\\-\sqrt{3} & 1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}3 & 0\\0 & 1\end{pmatrix}$

Wow! $\rho_{Mix1} = \rho_{Mix2}$

This ρ captures everything Alice can know

Def: A Matrix ρ is a <u>density matrix</u> (denmat) (or <u>density operator</u>) iff

a) $\mathrm{Tr}(\rho) = 1$

b) $\rho \geq 0$ (ρ is positive: i.e. $\forall |\phi\rangle$, $\langle\phi|\rho|\phi\rangle \geq 0$ $\Leftrightarrow$ ρ Hermitian & $\mathrm{eig}(\rho) \geq 0$)

<u>Claim #1</u>: $\rho = \sum_k \underset{\text{Probabilities}}{P_k} \, |\psi_k\rangle\langle\underset{\text{Normalized}}{\psi_k}|$

<u>Pf</u>:
- $$\begin{aligned} \mathrm{Tr}(\rho) &= \sum_k \mathrm{Tr}\left(P_k |\psi_k\rangle\langle\psi_k|\right) \\ &= \sum_k \mathrm{Tr}\left(P_k \underbrace{\langle\psi_k|\psi_k\rangle}_{1 \text{ since normalized}}\right) \\ &= \sum_k \mathrm{Tr}(P_k) \\ &= 1 \end{aligned}$$

- $$\sum_k P_k \langle\phi|\psi_k\rangle\langle\psi_k|\phi\rangle = \sum_k P_k |\langle\phi|\psi\rangle|^2 \geq 0$$

<u>Claim:</u> Any densmat ρ can be expressed as

$$\rho = \sum_k P_k |\psi_k\rangle\langle\psi_k|$$

for some $|\psi_k\rangle$, prob P_k

<u>Pf:</u> ρ is Hermitian, so $\exists\ \rho = \sum_k \lambda_k |k\rangle\langle k|$

eigenvalue eigenvector

<u>Def:</u> ρ is <u>pure</u> iff $\exists\ |\psi\rangle$ s.t. $\rho = |\psi\rangle\langle\psi|$

Otherwise, ρ is <u>mixed</u>

(Do not confuse mixed w/ superposition)

<u>Q:</u> $\rho = \sum P_k \rho_k$ (P_k prob) a densmat?
<u>A:</u> Yes.

<u>Q:</u> $\rho = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ a densmat?

<u>A:</u> No.

Lemma: Unravelings

$$\text{let } \rho = \sum P_k |\psi_k\rangle\langle\psi_k|$$

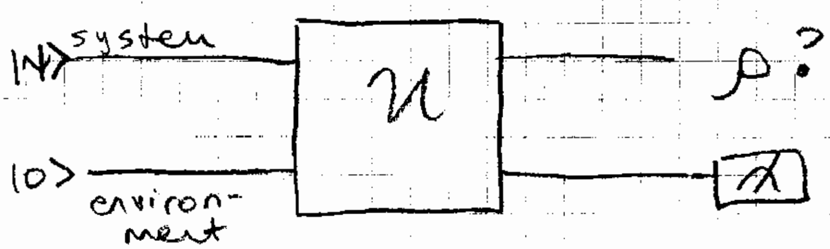
$$\text{Then } \rho = \rho' = \sum q_k |\varphi_k\rangle\langle\varphi_k|$$

$$\text{if } \sqrt{P_k}\,|\psi_k\rangle = \sum_j \mathcal{U}_{kj}\sqrt{q_j}\,|\varphi_j\rangle$$

with $\mathcal{U}_{kj}$ unitary

Why: Because it can't matter what basis B uses.

$\boxed{\text{② System-Environment}}$



$$|e\rangle|\psi\rangle \xrightarrow{\mathcal{U}} \mathcal{U}\,|e\rangle|\psi\rangle$$

Orthonormal basis
$|e_0\rangle\langle e_0|$
$|e_1\rangle\langle e_1|$
...

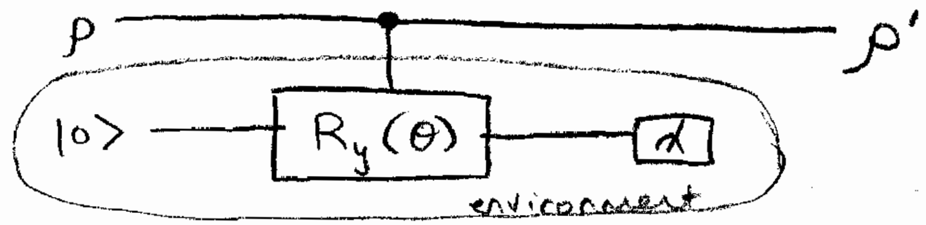
Final state is mixture:

$$\langle e_1|\mathcal{U}|e\rangle|\psi\rangle \oplus \langle e_2|\mathcal{U}|e\rangle|\psi\rangle \oplus \dots$$

$$\text{let } E_k := \langle e_k|\mathcal{U}|e\rangle$$

So mixture is $E_0|\psi\rangle \oplus E_1|\psi\rangle \oplus \dots \Rightarrow \rho = \sum_k E_k \rho_k E_k^\dagger$

<u>Example:</u>



$$\overset{\text{env sys}}{\mathcal{U}|00\rangle} = |00\rangle$$

$$\mathcal{U}|01\rangle = \left(\cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle\right)|1\rangle$$

$$E_0 = \overset{\text{env}}{\langle 0|}\,\mathcal{U}\,\overset{\text{env}}{|0\rangle} = |0\rangle\langle 0| + \cos(\theta/2)\,|1\rangle\langle 1|$$

$$E_1 = \overset{\text{env}}{\langle 1|}\,\mathcal{U}\,\overset{\text{env}}{|0\rangle} = \sin(\theta/2)\,|1\rangle\langle 1|$$

(Notation: let $\sqrt{p} := \cos\theta/2$)

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{p} \end{pmatrix}$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}$$

let $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

So $\rho' = \sum_k E_k \rho E_k^\dagger$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{p} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{p} \end{pmatrix} + \ldots$$

$\Rightarrow \rho = \begin{pmatrix} a & \sqrt{p}\,b \\ \sqrt{p}\,c & d \end{pmatrix}$ Note $\sqrt{p} \to 0$ as $\cos\theta/2 \to 0$

This is an example of <u>phase damping</u>

$\boxed{③ \text{ Quantum operations}}$

In general, what are the legal transformations
$\rho \to \mathcal{E}(\rho)$?

<u>Def:</u> Operation $\mathcal{E}$ is a valid quantum op iff

(A1) $\mathrm{Tr}(\mathcal{E}(\rho)) = 1$

(A2) $\mathcal{E}$ convex and linear

$$\mathcal{E}\left(\sum_k p_k \rho_k\right) = \sum_k p_k \mathcal{E}(\rho_k)$$

(A3) $\mathcal{E}$ is "completely positive"

a) if $\rho \geq 0$ then $\mathcal{E}(\rho) \geq 0$

b) $(\underset{\substack{\uparrow \\ \text{ref}}}{I_R} \otimes \underset{\substack{\uparrow \\ \text{quantum}}}{\mathcal{E}_Q})(\rho_{RQ}) \geq 0 \quad \forall \rho_{AB} \geq 0$

Why (A3b)?

Consider $\mathcal{E} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$$\sum_{jk} c_{jk} |j\rangle\langle k| \xrightarrow{\mathcal{E}} \sum_{jk} c_{jk} |k\rangle\langle j|$$

Is $\mathcal{E}$ a legal quantum op?

– (cont) This is positive: $\mathcal{E}(\rho) \geq 0 \quad \forall \rho \geq 0$

Consider $(I \otimes \mathcal{E}) \underbrace{\tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)}$

$$\frac{1}{2}\left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]$$

Take transpose of each of the 4 little matrices

$$= \begin{bmatrix} 1 & & & 0 \\ & 0 & 1 & \\ & 1 & 0 & \\ 0 & & & 1 \end{bmatrix}$$

⟵ Not a legal element! (Saw earlier)

– **Q:** Single qubit $\rho \xrightarrow{\mathcal{E}} \rho'$

How many DOFs describe $\mathcal{E}$? (Guesses: 4, 24, 16?)

A: 12

Q: For 2 qubits?

A: 240

$\boxed{\text{④ Operator Sum Representation}}$

Thm: let $\rho \in \mathcal{H}_1$ and $\mathcal{E}(\rho) \in \mathcal{H}_2$

Then $\mathcal{E}$ satisfies (A1), (A2), (A3) iff

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^{+}$$

where $\sum_k E_k^{+} E_k = 1$

E_k are op elements (Krauss ops) that map $\mathcal{H}_1 \to \mathcal{H}_2$

Pf ($\rightarrow$ dir):

(A2) $\mathcal{E}(\rho)$ obviously linear ✓
(don't worry about convexity)

(A1) $$\mathrm{Tr}(\mathcal{E}(\rho)) = \sum_k \mathrm{Tr}(E_k \rho E_k^\dagger) = \sum_k \mathrm{Tr}(\rho E_k^\dagger E_k) = \mathrm{Tr}(\rho) = 1$$

(A3) Let $|\psi_{RQ}\rangle \in \mathcal{H}_{RQ}$ s.t.

$$\sum_k \underbrace{\langle\psi|(I\otimes E_k)}_{\langle\phi_{RQ}|} P_{RQ} \underbrace{(I\otimes E_k)|\psi_{RQ}\rangle}_{|\phi_{RQ}\rangle}$$

Then $= \sum_k \langle\phi_{RQ}|P_{RQ}|\phi_{RQ}\rangle \geq 0$

Important Fact:

Let $E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{p} \end{bmatrix}$, $E_1 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}$ (Phase damping)

$\tilde{E}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\sqrt{\alpha}$, $\tilde{E}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\sqrt{1-\alpha}$

... behave exactly the same!

Important for stability, correcting phase errors.