LECTURE 16

- Readings: Section 5.1

Lecture outline

- Random processes
- Definition of the Bernoulli process
- Basic properties of the Bernoulli process
  - Number of successes
  - Distribution of interarrival times
  - The time of the k^{th} success

Random Processes: Motivation

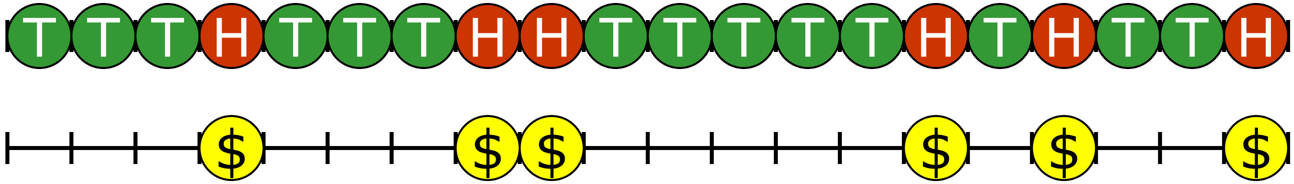
- Sequence of random variables: $X_1, X_2, \cdots$

Examples:

- Arrival example: Arrival of people to a bank.
- Queuing example: Length of a line at a bank.
- Gambler's ruin: The probability of an outcome is a function of the probability of other outcomes (Markov Chains).
- Engineering example: Signal corrupted with noise.

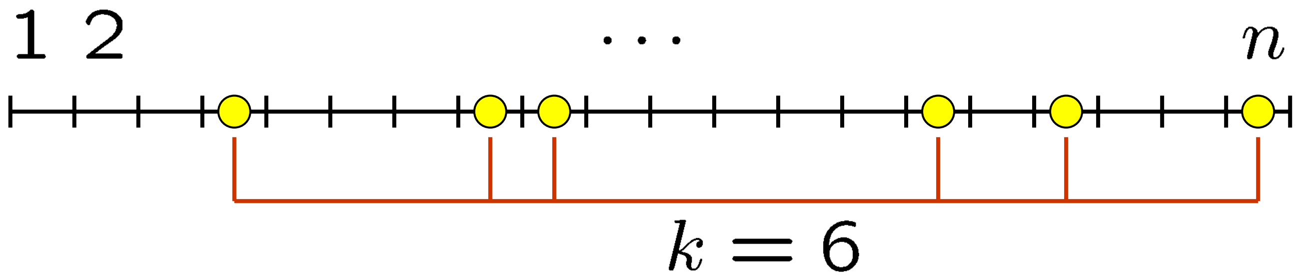
The Bernoulli Process

- A sequence of independent Bernoulli trials.
- At each trial:
  - $\mathbf{P}(\text{success}) = \mathbf{P}(X = 1) = p$
  - $\mathbf{P}(\text{failure}) = \mathbf{P}(X = 0) = 1 - p$



- **Examples**:
  - Sequence of ups and downs of the Dow Jones.
  - Sequence of lottery wins/losses.
  - Arrivals (each second) to a bank.

Number of successes S in n time slots



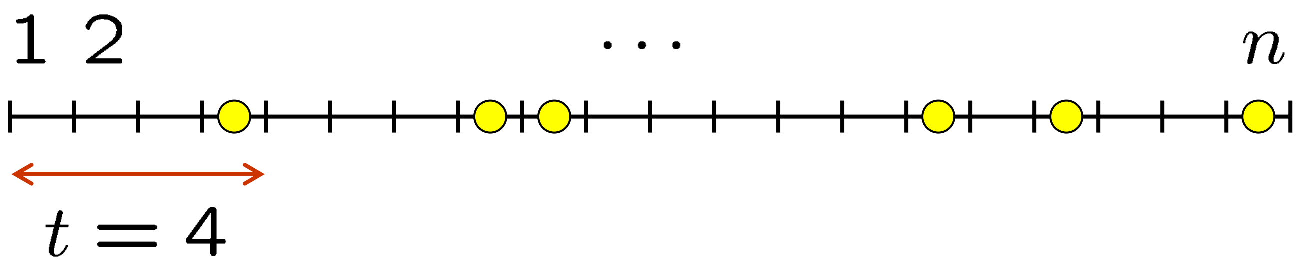
- $\mathrm{P}(S = k) = \binom{n}{k} p^k (1-p)^{n-k},$ **(Binomial)**

$$k = 0, 1, \cdots, n$$

- **Mean**: $\mathrm{E}[S] = np$

- **Variance**: $\mathrm{Var}(S) = np(1-p)$

Interarrival Times



- T_1 : number of trials until first success (inclusive).
- $\mathbf{P}(T_1 = t) = \; p(1-p)^{t-1},$ **(Geometric)**

$$t = 1, 2, \cdots$$

- **Memoryless property.**
- **Mean**: $\mathbf{E}[T_1] = \dfrac{1}{p}$
- **Variance**: $\mathsf{Var}(T_1) = \dfrac{1-p}{p^2}$

Fresh Start and Memoryless Properties

Fresh Start

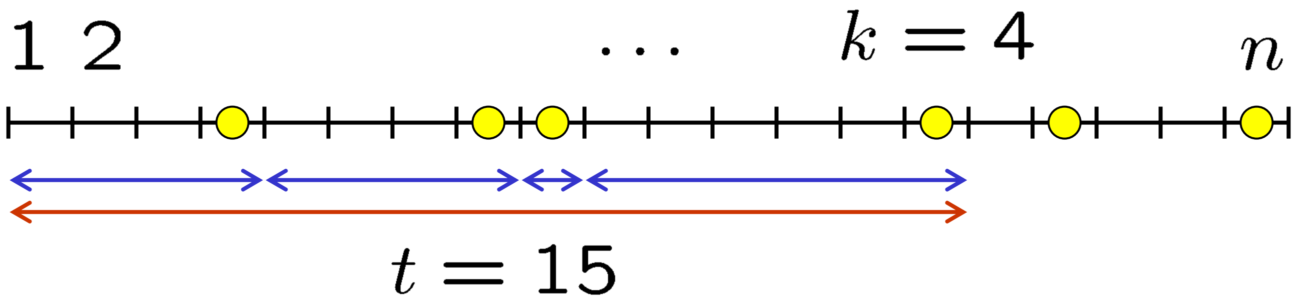
Given n, the future sequence $X_{n+1}, X_{n+2}, \ldots$ *is a also a Bernoull process and is independent of the past.*

Memorylessness

Suppose we observe the process for n times and no success occurred. Then the pmf of the remaining time for arrival is geometric.

$$\mathbf{P}(T - n = k | T > n) = p(1-p)^{k-1}$$

Time of the k^{th} Arrival



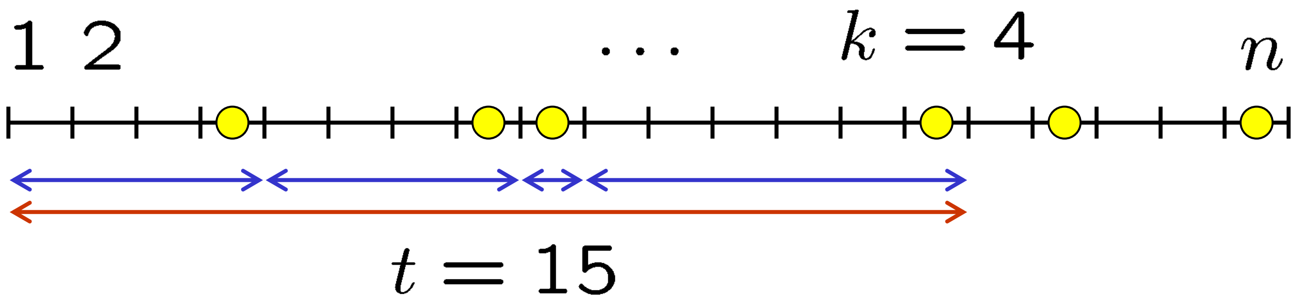
- Y_k : number of trials until k^{th} success (inclusive).

- $T_k = Y_k - Y_{k-1}$ $k = 2, 3, \dots$: kth interarrival time

- It follows that:

$$Y_k = T_1 + T_2 + \dots T_k$$

Time of the k^{th} Arrival



- Y_k : number of trials until k^{th} success (inclusive).
- **Mean**: $\mathbf{E}[Y_k] = \dfrac{k}{p}$
- **Variance**: $\text{Var}(Y_k) = \dfrac{k(1-p)}{p^2}$
- $\mathbf{P}(Y_k = t) = \binom{t-1}{k-1} p^k (1-p)^{t-k},$ **(Pascal)**

$$t = k, k+1, \cdots$$