

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
Department of Electrical Engineering & Computer Science  
**6.041/6.431: Probabilistic Systems Analysis**  
(Spring 2006)

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**Problem Set 9**  
**Topics: Bernoulli, Poisson**  
**Due: May 3rd, 2006**

1. A successful call occurs with probability  $p = \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$ .

- (a) Fred will give away his first sample on the third call if the first two calls are failures and the third is a success. Since the trials are independent, the probability of this sequence of events is simply

$$(1-p)(1-p)p = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

- (b) The event of interest requires failures on the ninth and tenth trials and a success on the eleventh trial. For a Bernoulli process, the outcomes of these three trials are independent of the results of any other trials and again our answer is

$$(1-p)(1-p)p = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

- (c) We desire the probability that  $L_2$ , the second-order interarrival time is equal to five trials. We know that  $p_{L_2}(l)$  is a Pascal PMF, and we have

$$p_{L_2}(5) = \binom{5-1}{2-1} p^2 (1-p)^{5-2} = 4 \cdot \left(\frac{1}{2}\right)^5 = \frac{1}{8}$$

- (d) Here we require the conditional probability that the experimental value of  $L_2$  is equal to 5, given that it is greater than 2.

$$\begin{aligned} p_{L_2|L_2>2}(5|L_2 > 2) &= \frac{p_{L_2}(5)}{P(L_2 > 2)} = \frac{p_{L_2}(5)}{1 - p_{L_2}(2)} \\ &= \frac{\binom{5-1}{2-1} p^2 (1-p)^{5-2}}{1 - \binom{2-1}{2-1} p^2 (1-p)^0} = \frac{4 \cdot \left(\frac{1}{2}\right)^5}{1 - \left(\frac{1}{2}\right)^2} = \frac{1}{6} \end{aligned}$$

- (e) The probability that Fred will complete at least five calls before he needs a new supply is equal to the probability that the experimental value of  $L_2$  is greater than or equal to 5.

$$\begin{aligned} P(L_2 \geq 5) &= 1 - P(L_2 \leq 4) = 1 - \sum_{l=2}^4 \binom{l-1}{2-1} p^2 (1-p)^{l-2} \\ &= 1 - \left(\frac{1}{2}\right)^2 - \binom{2}{1} \left(\frac{1}{2}\right)^3 - \binom{3}{1} \left(\frac{1}{2}\right)^4 = \frac{5}{16} \end{aligned}$$

- (f) Let discrete random variable  $F$  represent the number of failures before Fred runs out of samples on his  $m$ th successful call. Since  $L_m$  is the number of trials up to and including the  $m$ th success, we have  $F = L_m - m$ . Given that Fred makes  $L_m$  calls before he needs a new supply, we can regard each of the  $F$  unsuccessful calls as trials in another Bernoulli
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---

process with parameter  $r$ , where  $r$  is the probability of a success (a disappointed dog) obtained by

$$\begin{aligned} r &= \text{P(dog lives there} \mid \text{Fred did not leave a sample)} \\ &= \frac{\text{P(dog lives there AND door not answered)}}{1 - \text{P(giving away a sample)}} = \frac{\frac{1}{4} \cdot \frac{2}{3}}{1 - \frac{1}{2}} = \frac{1}{3} \end{aligned}$$

We define  $X$  to be a Bernoulli random variable with parameter  $r$ . Then, the number of dogs passed up before Fred runs out,  $D_m$ , is equal to the sum of  $F$  Bernoulli random variables each with parameter  $r = \frac{1}{3}$ , where  $F$  is a random variable. In other words,

$$D_m = X_1 + X_2 + X_3 + \cdots + X_F.$$

Note that  $D_m$  is a sum of a random number of independent random variables. Further,  $F$  is independent of the  $X_i$ 's since the  $X_i$ 's are defined in the conditional universe where the door is not answered, in which case, whether there is a dog or not does not affect the probability of that trial being a failed trial or not. From our results in class, we can calculate its expectation and variance by

$$\begin{aligned} E[D_m] &= E[F]E[X] \\ \text{var}(D_m) &= E[F]\text{var}(X) + (E[X])^2\text{var}(F), \end{aligned}$$

where we make the following substitutions.

$$\begin{aligned} E[F] &= E[L_m - m] = \frac{m}{p} - m = m. \\ \text{var}(F) &= \text{var}(L_m - m) = \text{var}(L_m) = \frac{m(1-p)}{p^2} = 2m. \\ E[X] &= r = \frac{1}{3}. \\ \text{var}(X) &= r(1-r) = \frac{2}{9}. \end{aligned}$$

Finally, substituting these values, we have

$$\begin{aligned} E[D_m] &= m \cdot \frac{1}{3} = \frac{m}{3} \\ \text{var}(D_m) &= m \cdot \frac{2}{9} + \left(\frac{1}{3}\right)^2 \cdot 2m = \frac{4m}{9} \end{aligned}$$

2. Define the following events:

Event R: Robber attempts robbery.  
Event S: Robbery is successful.

Define the following random variables:

$x$ : The number of days up to and including the first successful robbery.  
 $b$ : The number of candy bars stolen during a successful robbery.  
 $c$ : The number of days of rest after a successful robbery.

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Note that  $x$  is a geometric random variable with parameter  $\frac{3}{20}$  (the probability of a successful robbery on a given night). Also, it is given that  $b$  is uniform over  $\{1, 2, 3\}$  with probability  $\frac{1}{3}$  each, and that  $c$  is uniform over  $\{2, 4\}$  with probability  $\frac{1}{2}$  each.

- (a) Observe that since the probability that any given robbery attempt succeeds is  $\frac{3}{4}$ , the random variable  $k$  is geometric with parameter  $p = \frac{3}{4}$ . Thus,

$$p_k(k_o) = (1 - p)^{k_o-1} p = \left(\frac{1}{4}\right)^{k_o-1} \frac{3}{4} = \frac{3}{4^{k_o}} \quad \text{for } k_o = 1, 2, \dots$$

- (b) We will derive first the PMF and then the transform of  $d$ , the number of days up to and including the second successful robbery.

The PMF of  $d$  can be easily found by conditioning on  $c$ , the number of days the robber rests after the first successful robbery (which only takes on values 2 or 4):

$$p_d(d_o) = p_{d|c}(d_o | c = 2)P(c = 2) + p_{d|c}(d_o | c = 4)P(c = 4),$$

where

$$p_{d|c}(d_o | c = 2) = \begin{cases} \binom{d_o-3}{1} \left(\frac{3}{20}\right) \left(\frac{17}{20}\right)^{d_o-4} \left(\frac{3}{20}\right) & \text{if } d_o = 4, 5, 6, \dots \\ 0 & \text{otherwise.} \end{cases}$$

There must be at least one day before the rest period, since it follows a successful robbery; similarly, there must be at least one day after the rest period. Thus we can view the coefficient in the preceding formula as the number of ways to choose the beginning of a four-day period in a block of  $d_o$  days. Then we multiply the probability of the first success and the probability of  $d_o - 4$  failures and finally the probability of the second success at trial  $d_o$ .

Similarly, we have:

$$p_{d|c}(d_o | c = 4) = \begin{cases} \binom{d_o-5}{1} \left(\frac{3}{20}\right) \left(\frac{17}{20}\right)^{d_o-6} \left(\frac{3}{20}\right) & \text{if } d_o = 6, 7, 8, \dots \\ 0 & \text{otherwise.} \end{cases}$$

Also,  $P(c = 2) = P(c = 4) = \frac{1}{2}$ . So plugging into our expression for  $p_d(d_o)$ , we get

$$p_d(d_o) = \begin{cases} \frac{1}{2} \binom{d_o-3}{1} \left(\frac{3}{20}\right)^2 \left(\frac{17}{20}\right)^{d_o-4} & \text{if } d_o = 4, 5 \\ \frac{1}{2} \binom{d_o-3}{1} \left(\frac{3}{20}\right)^2 \left(\frac{17}{20}\right)^{d_o-4} + \frac{1}{2} \binom{d_o-5}{1} \left(\frac{3}{20}\right)^2 \left(\frac{17}{20}\right)^{d_o-6} & \text{if } d_o \geq 6 \\ 0 & \text{otherwise.} \end{cases}$$

Alternately, we can solve this problem using transforms. Note that  $d$  is the sum of three independent random variables:  $x_1$ , the number of days until the first successful robbery;  $c$ , the number of days of rest after the first success; and  $x_2$ , the number of days from the end of the rest period until the second successful robbery. Since  $d = x_1 + c + x_2$ , and  $x_1$ ,  $x_2$ , and  $c$  are mutually independent, we find that the transform of  $d$  is  $M_d(s) = [M_x(s)]^2 M_c(s)$ .

Now,  $x_1$  and  $x_2$  are (independent) geometric random variables with parameter  $\frac{3}{20}$  (the probability of a successful robbery). Again,  $c$  is equally likely to be 2 or 4. Thus we conclude that

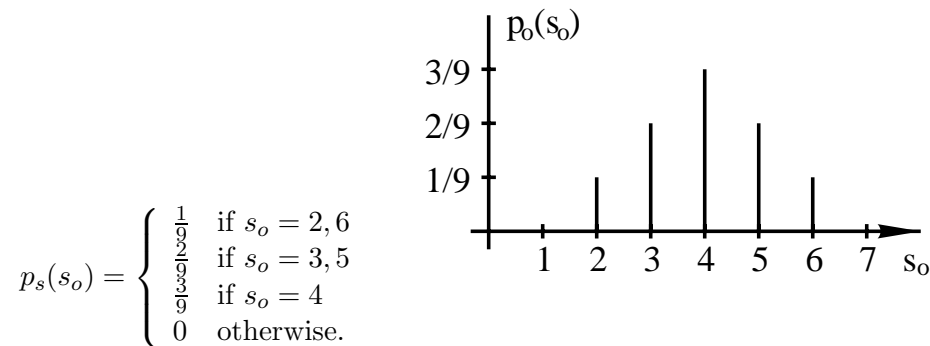
$$M_d(s) = [M_x(s)]^2 M_c(s) = \left[ \frac{\frac{3}{20} e^s}{1 - \frac{17}{20} e^s} \right]^2 \left[ \frac{1}{2} e^{2s} + \frac{1}{2} e^{4s} \right].$$


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---

- (c) Given a successful robbery, the PMF for  $y$  is  $p_y(y_o) = \frac{1}{3}$  for  $y_o = 1, 2, 3$ , and  $p_y(y_o) = 0$  otherwise. The total number of candybars collected in 2 successful robberies is  $s = y_1 + y_2$ , where  $y_1$  and  $y_2$  are independent and identically distributed as  $p_y(y_o)$ . Therefore, the PMF for  $s$  is



- (d) Observe that since the probability of a robbery attempt being successful is  $\frac{3}{4}$ , with the number of candy bars taken in a successful attempt equally likely to be 1, 2, or 3, we can view each attempt as resulting in  $b$  candy bars, with the following PMF for  $b$ :

$$p_b(b_o) = \begin{cases} \frac{1}{4} & \text{if } b_o = 0, 1, 2, 3, \\ 0 & \text{otherwise.} \end{cases}$$

Now, let  $t$ ,  $u$ , and  $v$  be the number of attempts that result in 1, 2, and 3 candy bars, respectively. Then if a total of  $q$  candy bars are stolen during the ten robbery attempts,  $0 \leq v \leq \frac{q}{3}$ ,  $0 \leq u \leq \frac{q-3v}{2}$ , and  $0 \leq t \leq q - 3v - 2u$ , with exactly  $10 - t - u - v$  attempts failing. Thus the PMF for  $q$  is

$$p_q(q_o) = \sum_{v=0}^{q_o/3} \sum_{u=0}^{\frac{q_o-3v}{2}} \sum_{t=0}^{q_o-3v-2u} \binom{10}{t, u, v, 10-t-u-v}, \quad \text{for } 0 \leq q_o \leq 30.$$

Of course, since  $q$  is the sum of ten independent values of  $b$ , the transform is simply  $M_q(s) = [M_b(s)]^{10} = (1 + e^s + e^{2s} + e^{3s})^{10}/4^{10}$ .

Observe that since  $q = b_1 + \dots + b_{10}$ ,  $E(q) = E(b_1) + \dots + E(b_{10}) = 10 \cdot E(b) = 15$ . Similarly, using the independence of the  $b_i$ 's, we see that  $\text{var}(q) = \text{var}(b_1) + \dots + \text{var}(b_{10}) = 10 \cdot \text{var}(b) = 10 \cdot \frac{5}{4} = \frac{50}{4}$ .

3. (a) i. Since it is implied that each second is independent from each other (i.e. if a mosquito landed on your neck last second, it doesn't affect the likelihood of one landing this or next second), the PMF for the time until the first mosquito lands is simply a geometric random variable. In this random variable, the "success" event is when a mosquito lands with probability 0.2, with the "failure" event being 0.8.

$$p_T(t) = \begin{cases} (.8)^{t-1}(.2) & t \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
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**6.041/6.431: Probabilistic Systems Analysis**  
(Spring 2006)

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- ii. The expected time of the geometric PMF with parameter  $p$  is  $\frac{1}{p}$ , so the expected time until the first mosquito lands on you is  $\frac{1}{2} = 5$  seconds.
  - iii. The scenario in which mosquitoes independently land on you each second can be modeled as a Bernoulli process, with each Bernoulli trial being the event that a mosquito lands on you on a given second. Because the Bernoulli process is memoryless, it doesn't matter whether or not you were bitten for the first one, ten, or two hundred seconds. Thus, the expected time from  $T = 10$  is identical to the answer in the previous part,  $\frac{1}{2} = 5$  seconds.
- (b)
- i. Because the PDF that models the time until the first mosquito arrives is exponential, the expected time until it lands is  $\frac{1}{2} = 5$  seconds.
  - ii. It has been previously shown that the exponential PDF exhibits the memorylessness property. In other words, looking at an exponential PDF from some future time  $\neq 0$  will still yield an exponential PDF with the same parameter. Thus, the expected time from  $T = 10$  is identical to the answer in the previous part,  $\frac{1}{2} = 5$  seconds.
4. We could find an exact value by using the binomial probability mass function. A reasonable, and much more efficient method is to use the Poisson approximation to the binomial, which tells us that for a binomial random variable with parameters  $n$  and  $p$ , we have:

$$\mathbf{P}(k \text{ successes}) \approx \frac{\lambda^k}{k!} e^{-\lambda}$$

where  $\lambda = np$ . The desired probability is

$$\mathbf{P}(2 \text{ or more fatalities}) = 1 - \mathbf{P}(0 \text{ or } 1 \text{ fatality}) = 1 - \frac{2^0}{0!} e^{-2} - \frac{2^1}{1!} e^{-2} = .594$$

5. We have a Poisson process with an average arrival rate  $\lambda$  which is equally likely to be either 2 or 4. Thus,

$$\mathbf{P}(\lambda = 2) = \mathbf{P}(\lambda = 4) = \frac{1}{2}.$$

We observe the process for  $t$  time units and observe  $k$  arrivals. The conditional probability that  $\lambda = 2$  is, by definition

$$\mathbf{P}(\lambda = 2 \mid k \text{ arrivals in time } t) = \frac{\mathbf{P}(\lambda = 2 \text{ and } k \text{ arrivals in time } t)}{\mathbf{P}(k \text{ arrivals in time } t)}.$$

Now, we know that

$$\mathbf{P}(\lambda = 2 \text{ and } k \text{ arrivals in time } t) = \mathbf{P}(k \text{ arrivals in time } t \mid \lambda = 2) \cdot \mathbf{P}(\lambda = 2) = \frac{(2t)^k e^{-2t}}{k!} \cdot \frac{1}{2}.$$

Similarly,

$$\mathbf{P}(\lambda = 2 \text{ and } k \text{ arrivals in time } t) = \frac{(4t)^k e^{-4t}}{k!} \cdot \frac{1}{4}.$$

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
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(Spring 2006)

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Thus,

$$\mathbf{P}(\lambda = 2 \mid k \text{ arrivals in time } t) = \frac{\frac{(2t)^k e^{-2t}}{k!} \left(\frac{1}{2}\right)}{\frac{(2t)^k e^{-2t}}{k!} \left(\frac{1}{2}\right) + \frac{(4t)^k e^{-4t}}{k!} \left(\frac{1}{2}\right)} = \frac{(2t)^k e^{-2t}}{(2t)^k e^{-2t} + (4t)^k e^{-4t}} = \frac{1}{1 + 2^k e^{-2t}}.$$

To check whether this answer is reasonable, suppose  $t$  is large and  $k = 2t$  (observed arrival rate equals 2). Then,  $\mathbf{P}(\lambda = 2 \mid k \text{ arrivals in time } t)$  approaches 1 as  $t$  goes to  $\infty$ . Similarly, if  $t$  is large and  $k = 4t$  (observed arrival rate equals 4), then,  $\mathbf{P}(\lambda = 2 \mid k \text{ arrivals in time } t)$  approaches 0 as  $t$  goes to  $\infty$ .

G1<sup>†</sup>. Problem 5.15 see online solutions