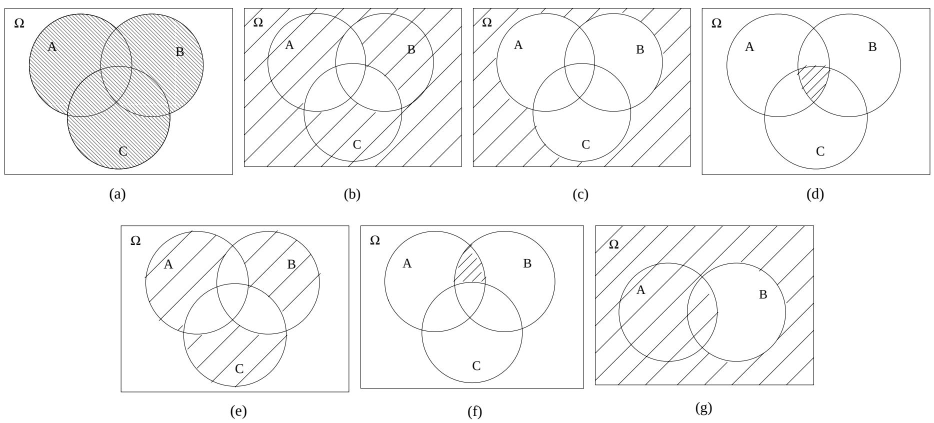
Problem Set 1: Solutions
Due: February 15, 2006

1. (a) $A \cup B \cup C$
   (b) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C) \cup (A^c \cap B^c \cap C^c)$
   (c) $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$
   (d) $A \cap B \cap C$
   (e) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$
   (f) $A \cap B \cap C^c$
   (g) $A \cup (A^c \cap B^c)$



2. (a) We have already proved in lecture and in the course notes that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Rearranging, we get

$$P(A \cap B) = P(A) + P(B) - P(A \cup B).$$

Since $(A \cup B)$ is always a subset of Ω, the universal event, therefore, $P(A \cup B) \leq P(\Omega)$ and

$$P(A \cap B) \geq P(A) + P(B) - P(\Omega).$$

Finally, by the normalization axiom, $P(\Omega) = 1$ and

$$P(A \cap B) \geq P(A) + P(B) - 1.$$

(b) We begin by writing

$$\begin{aligned} P(A \text{ or } B, \text{ but not both}) &= P((A^c \cap B) + (A \cap B^c)) \\ &= P(A^c \cap B) + P(A \cap B^c), \end{aligned}$$

where the last equality is from the additivity axiom. Next, we know that $B = (A^c \cap B) \cup (A \cap B)$ and $(A^c \cap B) \cap (A \cap B) = \emptyset$ so that we may apply the additivity axiom to get

$$P(B) = P(A^c \cap B) + P(A \cap B).$$

With rearrangement, this becomes

$$P(A^c \cap B) = P(B) - P(A \cap B).$$

By symmetry, we also have

$$P(B^c \cap A) = P(A) - P(A \cap B).$$

So plugging in for $P(A^c \cap B)$ and $P(B^c \cap A)$, we get

$$\begin{aligned} P(A \text{ or } B, \text{ but not both}) &= P(B) - P(A \cap B) + P(A) - P(A \cap B) \\ &= P(A) + P(B) - 2P(A \cap B). \end{aligned}$$

3. (a) We are given $P(A) = 3/7$, $P(B \cap C) = 0$ and $P(A \cap B \cap C) = 0$. Using De Morgan's laws, we know $(B^c \cup C^c)^c = B \cap C$. Therefore

$$P(A \cup (B^c \cup C^c)^c) = P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap (B \cap C)) = \boxed{3/7}.$$

(b) We are given $P(A) = 1/2$, $P(B \cap C) = 1/3$ and $P(A \cap C) = 0$. Therefore, again applying De Morgan's laws,

$$P(A \cup (B^c \cup C^c)^c) = P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap (B \cap C)) = \boxed{5/6}$$

where we deduce $A \cap B \cap C = \emptyset$ (and thus $P(A \cap B \cap C) = 0$) because $A \cap C = \emptyset$ and $A \cap B \cap C \subseteq A \cap C$.

(c) We are given $P(A^c \cap (B^c \cup C^c)) = 0.65$ and De Morgan's laws imply $(A^c \cap (B^c \cup C^c))^c = A \cup (B^c \cup C^c)^c$, which is the event of interest. Therefore

$$P(A \cup (B^c \cup C^c)^c) = 1 - P(A^c \cap (B^c \cup C^c)) = \boxed{0.35}$$

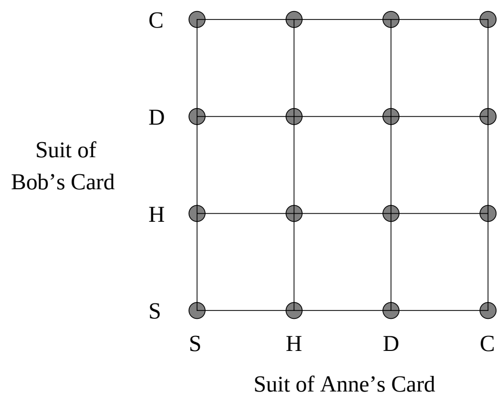
4. We could have a two-dimensional sample space containing 52^2 points, where each axis represents a particular card. However, this sample space would be finer grain than necessary to determine the desired probabilities.

For parts a) and c), we have a sample space of 169 points representing the 169 possible outcomes.

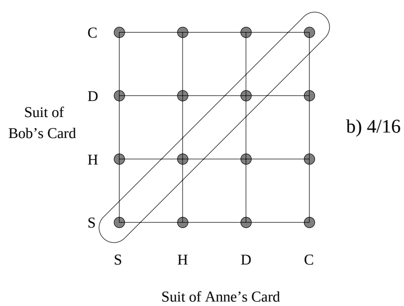
Define event B to be when Bob draws an ace, event A to be when Anne draws an ace. Then we know that

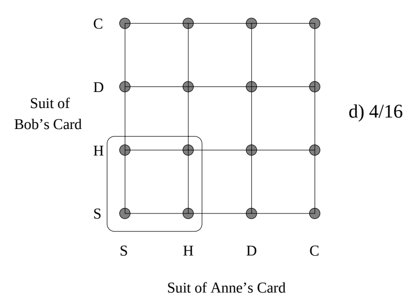
$P(A) = P(B) = \frac{1}{13}$
$P(A \cap B) = \frac{1}{169}$
P(at least one card is an ace) $= P(A \cup B) = P(A) + P(B) - P(A \cap B) = \boxed{\frac{25}{169}}$
P(neither card is an ace) = 1- P(at least one card is an ace) = $\boxed{\frac{144}{169}}$

For parts b) and d), since we are only interested in the suits of the cards, we represent the sample space as the following 16 points. The horizontal axis represents the suit of Anne's card, and the vertical axis represents the suit of Bob's card. Each of the points is equally likely; therefore, the probability of any particular point occurring is $\frac{1}{16}$.



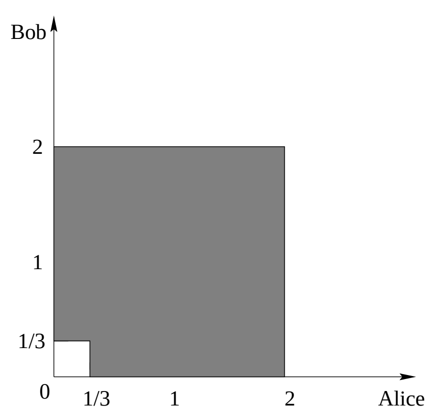
The probabilities requested can be determined by counting the number of points satisfying each condition and dividing the total by 16, as shown in the figures below.





5. **P(B)**

The shaded area in the following figure is the union of Alice's pick being greater than 1/3 and Bob's pick being greater than 1/3.

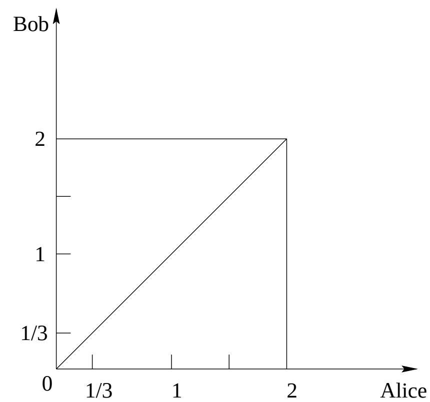


$$\begin{aligned} P(B) &= 1 - P(\textit{both numbers are smaller than } 1/3) \\ &= 1 - \frac{\textit{Area of small square}}{\textit{Total sample area}} \end{aligned}$$

$$
\begin{aligned}
&= 1 - \frac{1/3 * 1/3}{4} \\
&= 1 - 1/36 \\
&= \boxed{35/36}
\end{aligned}
$$

P(C)

In the following figure, the line x = y represents the set of points where two numbers are equal.

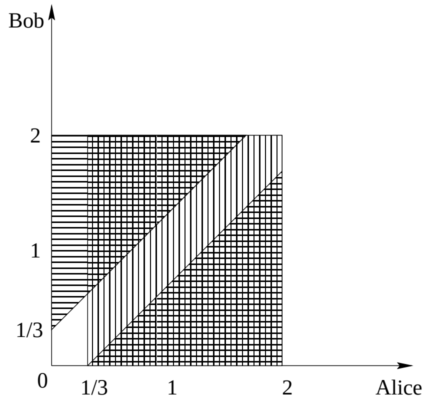


The line has an area of 0. Thus,

$$
\begin{aligned}
P(C) &= \frac{Area\ of\ line}{Total\ sample\ area} \\
&= \frac{0}{4} \\
&= \boxed{0}
\end{aligned}
$$

P($A \cap D$)

Overlapping the diagrams we would get for P(A) and P(D),



$$\begin{aligned} P(A \cap D) &= \frac{\textit{double shaded area}}{\textit{Total sample area}} \\ &= \frac{5/3 * 5/3 * 1/2 + 4/3 * 4/3 * 1/2}{4} \\ &= \frac{25/18 + 16/18}{4} \\ &= \boxed{41/72} \end{aligned}$$

6. We begin by enumerating the sample space Ω and identifying the *relative* probabilities of all outcomes, as shown in the table below, where $p \in [0, 1]$ will need to be determined.

Die 1	Die 2	Product	P(Product)
1	1	1	p
1	2	2	2p
1	3	3	3p
1	4	4	4p
2	1	2	2p
2	2	4	4p
2	3	6	6p
2	4	8	8p
3	1	3	3p
3	2	6	6p
3	3	9	9p
3	4	12	12p
4	1	4	4p
4	2	8	8p
4	3	12	12p
4	4	16	16p
		Total	100p

$$P(\Omega) = 1 = 100p \quad \Rightarrow \quad p = \frac{1}{100} = 0.01$$

(a) Let set A indicate the event that the product is even. Then,

$$P(A) = 2p + 4p + 2p + 4p + 6p + 8p + 6p + 12p + 4p + 8p + 12p + 16p = 84p = \boxed{0.84}$$

(b) Let set B indicate the event of rolling a 2 and 3. Then,

$$P(B) = P(2,3) + P(3,2) = 6p + 6p = 12p = \boxed{0.12}$$

G1[†]. (a) Define event E $= A \cup B$. Then $E \cap C = (A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
Therefore $P(E \cap C) = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$

$$\begin{aligned} P(A \cup B \cup C) &= P(E \cup C) \\ &= P(E) + P(C) - P(E \cap C) \\ &= P(A \cup B) + P(C) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \\ &= P(A) + P(B) + P(C) - P(A \cap C) - P(B \cap C) - P(A \cap B) + P(A \cap B \cap C) \end{aligned}$$

(b) We will apply an inductive argument.

Base Case:
$P(A_1) = P(A_1)$
Inductive Step:
Assume $P(\cup_{k=1}^{n-1} A_k) = P(A_1) + P(A_1^c \cap A_2) + P(A_1^c \cap A_2^c \cap A_3)$
$+ \cdots + P(A_1^c \cap \cdots \cap A_{n-2}^c \cap A_{n-1})$.

$$\begin{aligned} P(\cup_{k=1}^{n} A_k) &= P(\cup_{k=1}^{n-1} A_k \cup A_n) \\ &= P(A_1) + P(A_1^c \cap A_2) + P(A_1^c \cap A_2^c \cap A_3) + \cdots + P(A_1^c \cap \cdots \cap A_{n-2}^c \cap A_{n-1}) \\ &\quad + (P(A_n) - P(\cup_{k=1}^{n-1} A_k \cap A_n)) \\ &= P(A_1) + P(A_1^c \cap A_2) + P(A_1^c \cap A_2^c \cap A_3) + \cdots + P(A_1^c \cap \cdots \cap A_{n-2}^c \cap A_{n-1}) \\ &\quad + P((\cup_{k=1}^{n-1} A_k)^c \cap A_n) \\ &= P(A_1) + P(A_1^c \cap A_2) + P(A_1^c \cap A_2^c \cap A_3) + \cdots + P(A_1^c \cap \cdots \cap A_{n-1}^c \cap A_n). \end{aligned}$$

[†]Required for 6.431; optional for 6.041