FEM for the Poisson Problem in $\mathbb{R}^2$

April 14 & 16, 2003

Model Problem

Formulations

Strong Formulation

Find u such that

$$-\nabla^2 u = f \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \Gamma$$

for Ω a *polygonal domain*.

N1

Model Problem

Find $u = \arg\min_{w \in X} \underbrace{\tfrac{1}{2}a(w,w) - \ell(w)}_{J(w)}$;

or find $u \in X$ such that

$$a(u,v) = \ell(v), \forall v \in X \ ;$$

Model Problem

Formulations

...Minimization/Weak Formulations

where

$$X = \{v \in H^1(\Omega) \mid v|_\Gamma = 0\} \equiv H_0^1(\Omega) \ ,$$

$$a(w,v) = \int_\Omega \nabla w \cdot \nabla v \, dA \qquad \text{SPD} \ ,$$

$$\ell(v) = \text{`} \int_\Omega f \, v \, dA \text{'} \qquad \text{bounded} \ .$$

Model Problem

Regularity

In general, $\|u\|_{H^1(\Omega)} \leq C\|\ell\|_{H^{-1}(\Omega)}$.

If $f \in L^2(\Omega)$ *and* Ω is convex,

$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)};$$

N2

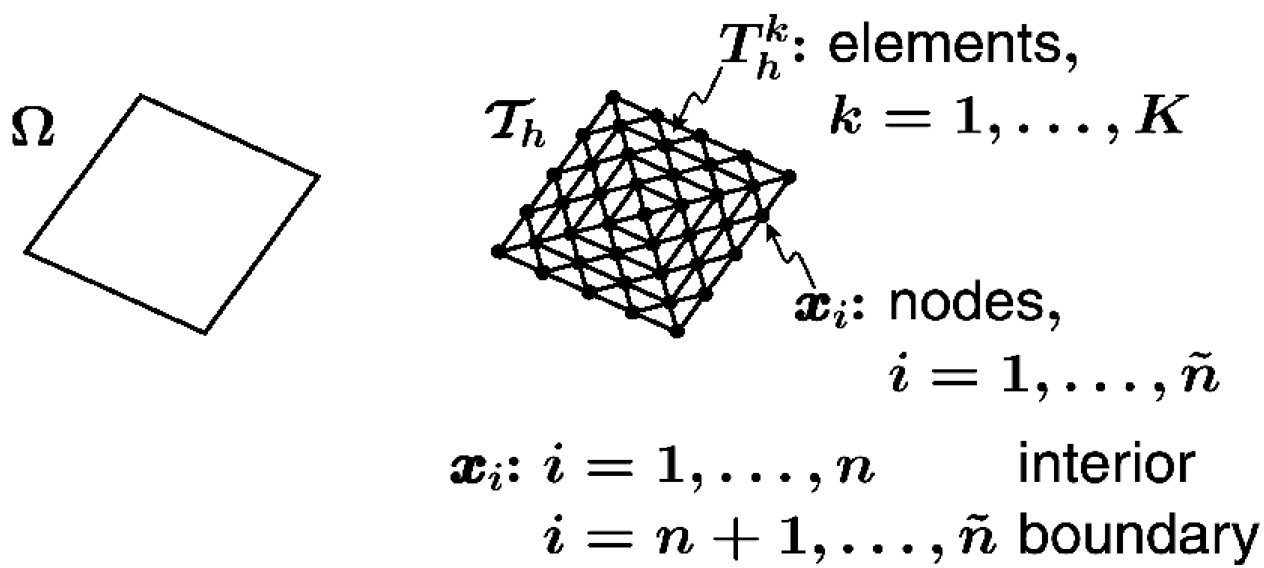
important for *convergence rate*.

Finite Element Discretization

Triangulation

N3

$$\overline{\Omega} = \bigcup_{T_h \in \mathcal{T}_h} \overline{T}_h$$

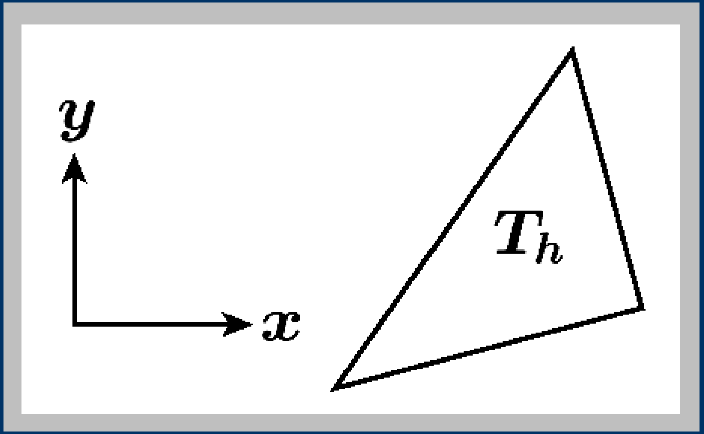


Finite Element Discretization

Approximation

Space (Linear Elements)

$$X_h = \{ \underbrace{v \in X}_{\substack{v|_\Gamma = 0,\\ v \in C^0(\Omega)}} \ | \ v|_{T_h} \in \mathbb{P}_1(T_h), \ \forall \, T_h \in \mathcal{T}_h \}$$



$$\mathbb{P}_1(T_h): \ v|_{T_h} = c_0 + \underbrace{c_x}_{v_x} x + \underbrace{c_y}_{v_y} y, \qquad c, c_x, c_y \in \mathbb{R}$$

Finite Element Discretization

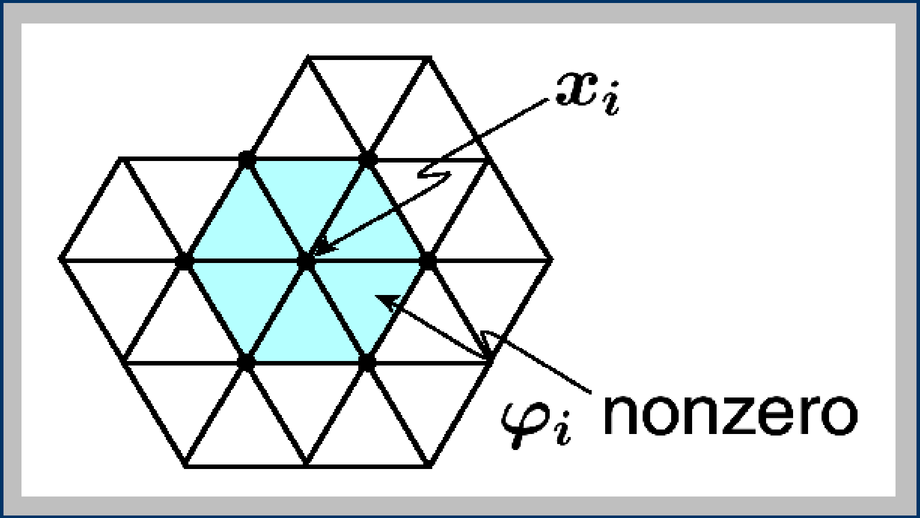
Approximation

Basis (Nodal)...

$X_h = \text{span}\,\{\varphi_1, \ldots, \varphi_n\}$:

$$\varphi_i \in X_h, \quad \varphi_i\,(x_j) = \delta_{ij}, \quad 1 \le i, j \le n.$$

Support of φ_i:



Finite Element Discretization

Approximation

...Basis (Nodal)

Nodal interpretation: $v \in X_h$,

$$v = \sum_{i=1}^{n} v_i \, \varphi_i(x) \; ;$$

$$v(x_j) = \sum_{i=1}^{n} v_i \, \varphi_i \, (x_j) = \sum_{i=1}^{n} v_i \, \delta_{ij} \Rightarrow \boxed{v_j = v(x_j)} \; .$$

Finite Element Discretization

“Projection”

Rayleigh-Ritz or Galerkin

Rayleigh-Ritz:

$$u_h = \arg\min_{w \in X_h} \underbrace{\tfrac{1}{2} a(w,w) - \ell(w)}_{J(w)}$$

Galerkin: $u_h \in X_h$ satisfies

$$a(u_h, v) = \ell(v), \quad \forall\, v \in X_h\ .$$

Finite Element Discretization

Discrete Equations

General Case

Let $u_h(x) = \sum_{j=1}^{n} u_{h\,j}\, \varphi_j(x)$; $v = \varphi_i(x)$, $i = 1, \ldots, n$:

$$\boxed{\underline{A}_h\, \underline{u}_h = \underline{F}_h} \qquad \underline{u}_h \in \mathbb{R}^n$$

$A_{h\,ij} = a(\varphi_i, \varphi_j)$, $1 \le i, j \le n$,

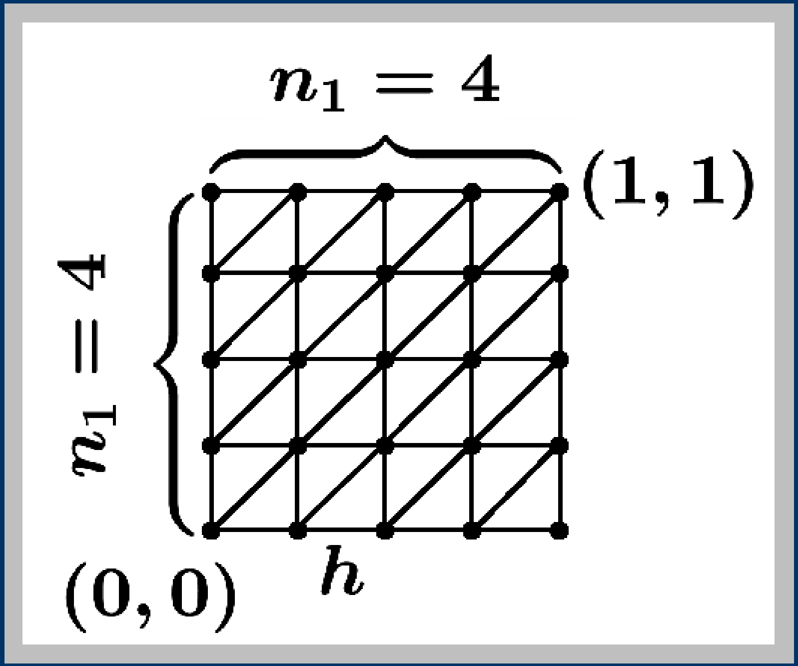
$F_{h\,i} = \ell(\varphi_i)$, $1 \le i \le n$.

Finite Element Discretization

Discrete Equations

Particular Illustrative Case...

Uniform Mesh:



$$K = 2n_1^2$$

$$\tilde{n} = (n_1 + 1)^2$$

$$n = (n_1 - 1)^2$$

$$h = 1/n_1$$

Finite Element Discretization

Discrete Equations

...Particular Illustrative Case...

Expression for $\underline{A}_h$:

$$a(w,v) = \int_\Omega \nabla w \cdot \nabla v \, dA = \int_\Omega w_x v_x + w_y v_y \, dA$$

$$\Downarrow$$

$$A_{h\,ij} = a(\varphi_i, \varphi_j) = \int_\Omega \frac{\partial \varphi_i}{\partial x}\frac{\partial \varphi_j}{\partial x} + \frac{\partial \varphi_i}{\partial y}\frac{\partial \varphi_j}{\partial y} \, dA$$

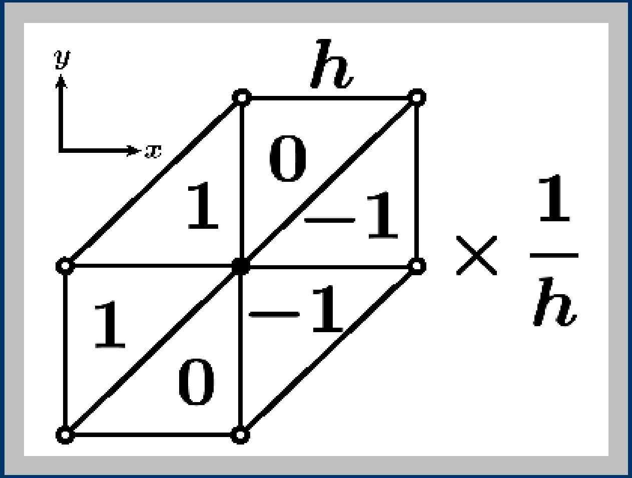
$$1 \le i, j \le n.$$

Finite Element Discretization

Discrete Equations

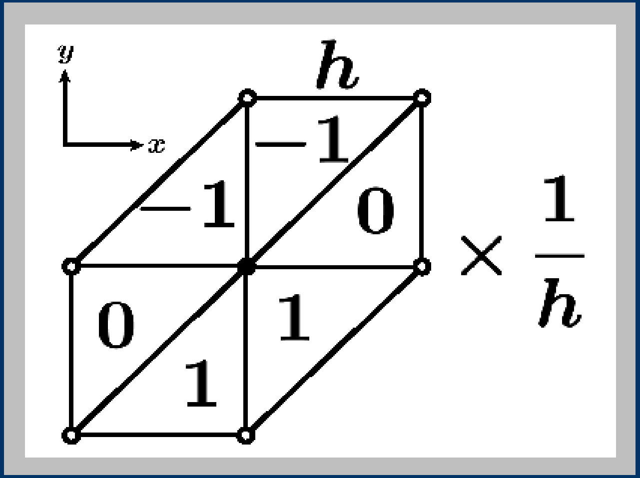
...Particular Illustrative Case...

Derivatives of φ_i: $\bullet x_i$



$\partial \varphi_i / \partial x$

(piecewise constant)



$\partial \varphi_i / \partial y$

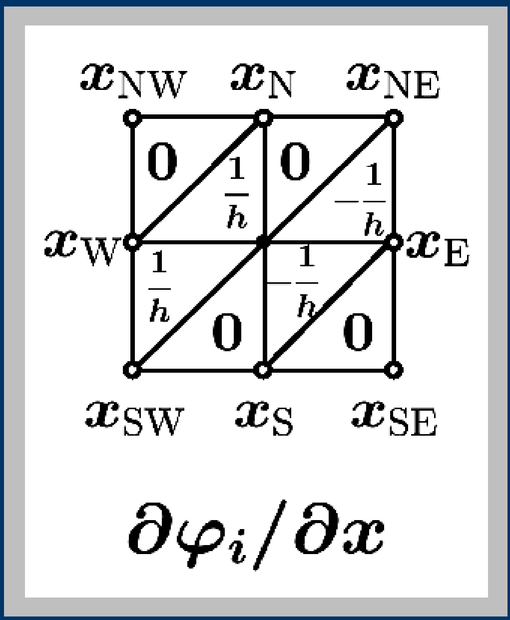
(piecewise constant)

Finite Element Discretization

Discrete Equations

...Particular Illustrative Case...

Evaluation of $\int_\Omega (\partial\varphi_i/\partial x)\,(\partial\varphi_j/\partial x)\,dA$ • x_i



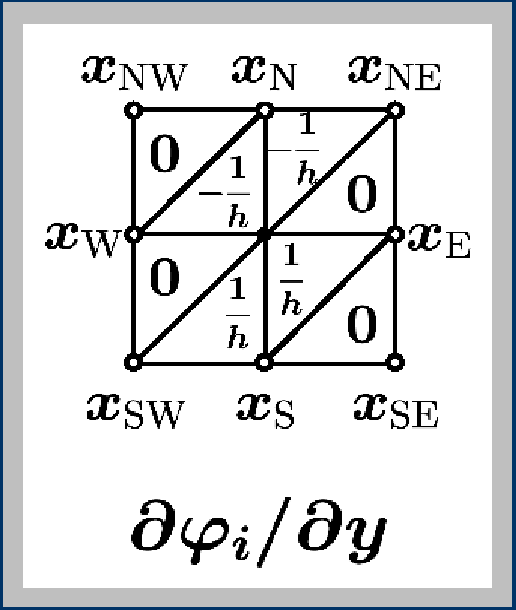
$$\int_\Omega \frac{\partial\varphi_i}{\partial x}\begin{Bmatrix}\partial\varphi_N/\partial x\\ \partial\varphi_{NE}/\partial x\\ \partial\varphi_E/\partial x\\ \partial\varphi_{SE}/\partial x\\ \partial\varphi_S/\partial x\\ \partial\varphi_{SW}/\partial x\\ \partial\varphi_W/\partial x\\ \partial\varphi_{NW}/\partial x\\ \partial\varphi_i/\partial x\end{Bmatrix} dA = \begin{Bmatrix}0\\ 0\\ -2/h^2\\ 0\\ 0\\ 0\\ -2/h^2\\ 0\\ 4/h^2\end{Bmatrix}\frac{h^2}{2}$$

Finite Element Discretization

Discrete Equations

...Particular Illustrative Case...

Evaluation of $\int_\Omega (\partial\varphi_i/\partial y)\,(\partial\varphi_j/\partial y)\,dA$ $\bullet\ x_i$



$$\int_\Omega \frac{\partial\varphi_i}{\partial y}\begin{Bmatrix}\partial\varphi_N/\partial y\\ \partial\varphi_{NE}/\partial y\\ \partial\varphi_E/\partial y\\ \partial\varphi_{SE}/\partial y\\ \partial\varphi_S/\partial y\\ \partial\varphi_{SW}/\partial y\\ \partial\varphi_W/\partial y\\ \partial\varphi_{NW}/\partial y\\ \partial\varphi_i/\partial y\end{Bmatrix} dA = \begin{Bmatrix}-2/h^2\\ 0\\ 0\\ 0\\ -2/h^2\\ 0\\ 0\\ 0\\ 4/h^2\end{Bmatrix}\frac{h^2}{2}$$

Finite Element Discretization

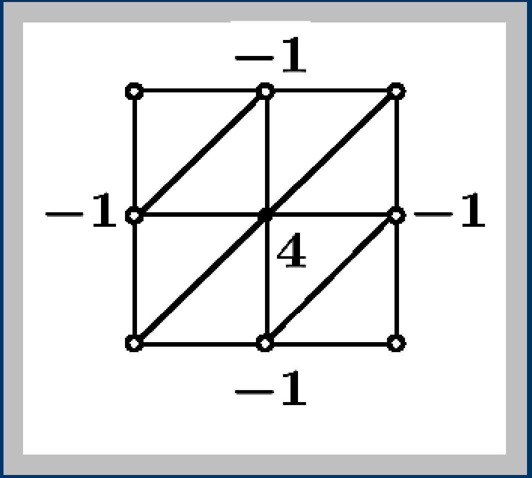
Discrete Equations

...Particular Illustrative Case

Summary

$\bullet x_i$

Nonzero entries of row i of $\underline{A}_h$:



identical to finite differences.

Recall $|||v|||^2 \equiv a(v,v) = |v|^2_{H^1(\Omega)} = \int_\Omega |\nabla v|^2 \, dA$.

Then

$$|||e||| = \inf_{w_h \in X_h} |||u - w_h||| \qquad (e = u - u_h) \; ;$$

u_h is the *projection* of u on X_h

in the energy norm.

Theoretical Analysis

General Results

H^1 Norm

Recall $\|v\|^2_{H^1(\Omega)} = \int_\Omega |\nabla v|^2 + v^2 \, dA$.

Then

$$\|e\|_{H^1(\Omega)} \le \left(1 + \frac{\beta}{\alpha}\right) \inf_{w_h \in X_h} \|u - w_h\|_{H^1(\Omega)} ;$$

α: coercivity constant (> 0);

β: continuity constant ($= 1$).

Theoretical Analysis

Particular Results

H^1 and L^2 Norms

For $f \in L^2(\Omega)$ and Ω convex,

$$|||e||| \leq C\, h\, \|u\|_{H^2(\Omega)}\ ;$$

$$\|e\|_{H^1(\Omega)} \leq C\, h\, \|u\|_{H^2(\Omega)}\ ;$$

and

$$\|e\|_{L^2(\Omega)} \leq C\, h^2\, \|u\|_{H^2(\Omega)}\ .$$

N4

Theoretical Analysis

Particular Results

Output Functionals

Recall $s = \ell^O(u) + c^O$, $s_h = \ell^O(u_h) + c^O$.

For $f \in L^2(\Omega)$ and Ω convex,

if $\ell^O \in H^{-1}(\Omega)$, $|s - s_h| = |\ell^O(e)| \leq C\, h\, \|u\|_{H^2(\Omega)}$;

if $\ell^O \in L^2(\Omega)$, $\quad |s - s_h| = |\ell^O(e)| \leq C\, h^2\, \|u\|_{H^2(\Omega)}$.

Implementation

Four steps:

A Proto-Problem;

Elemental Quantities;

Assembly;

Boundary Conditions;

and Numerical Quadrature.

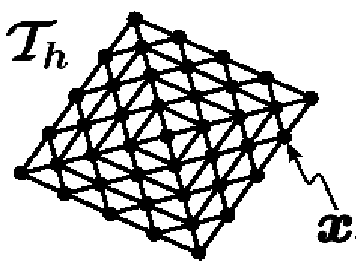
Implementation

A Proto-Problem

Space and Basis

Let $\tilde{X}_h = \{ v \in H^1(\Omega) \mid v|_{T_h} \in \mathbb{P}_1(T_h), \ \forall \, T_h \in \mathcal{T}_h \}$

$$= \text{span}\,\{\varphi_1, \ldots, \varphi_n, \ \varphi_{n+1}, \ldots, \varphi_{\tilde{n}}\}$$



$\boldsymbol{x}_i$: nodes,
$i = 1, \ldots, \tilde{n}$

$\boldsymbol{x}_i$: $i = 1, \ldots, n$ interior
$i = n+1, \ldots, \tilde{n}$ boundary

Implementation

A Proto-Problem

Statement

"Find" $\tilde{u}_h \in \tilde{X}_h$ such that

$$a(\tilde{u}_h, v) = \ell(v), \qquad \forall v \in \tilde{X}_h \ .$$

We never actually solve this problem;
it serves only as a convenient pre-processing step.

Implementation

A Proto-Problem

Discrete Equations

$$\underline{\tilde{A}}_h \, \underline{\tilde{u}}_h = \underline{\tilde{F}}_h \qquad\qquad \tilde{u}_h(\boldsymbol{x}) = \sum_{i=1}^{\tilde{n}} \tilde{u}_{h\,i}\, \varphi_i(\boldsymbol{x})$$

$$\tilde{A}_{h\,ij} = a(\varphi_i, \varphi_j) = \int_\Omega \frac{\partial \varphi_i}{\partial x}\frac{\partial \varphi_j}{\partial x} + \frac{\partial \varphi_i}{\partial y}\frac{\partial \varphi_j}{\partial y}\, dA$$

$$1 \leq i,\ j \leq \tilde{n}\ ;$$

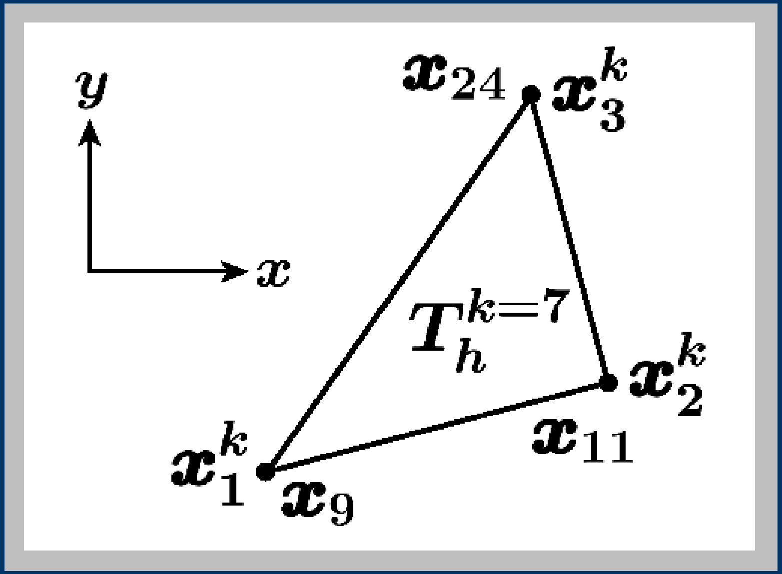
$$\tilde{F}_{h\,i} = \ell(\varphi_i) \left(= \int_\Omega f \varphi_i \right),\ 1 \leq i \leq \tilde{n}\ .$$

Implementation

Elemental Quantities

Local Definitions...

Local Nodes



$\mathbf{Area}^k$: area of T_h^k.

$\boldsymbol{x}_1^k, \boldsymbol{x}_2^k, \boldsymbol{x}_3^k$: local nodes in element T_h^k,
corresponding to global nodes $\boldsymbol{x}_9, \boldsymbol{x}_{11}, \boldsymbol{x}_{24}$ (say).

Implementation

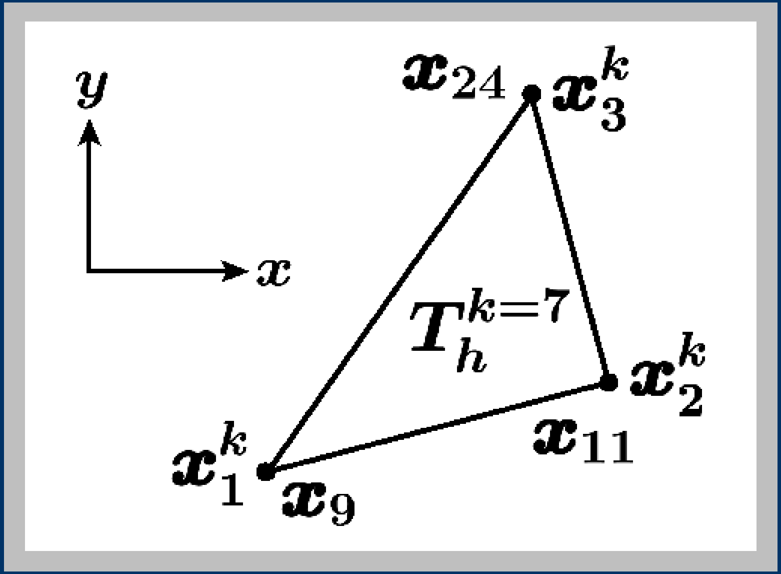
Elemental Quantities

...Local Definitions...

Local Basis Functions $\mathcal{H}_\alpha^k, \alpha = 1, 2, 3$:

$$\mathcal{H}_\alpha^k \in \mathbb{P}_1(T_h^k)$$

$$\mathcal{H}_\alpha^k(\boldsymbol{x}_\beta^k) = \delta_{\alpha\beta}$$



$$\mathcal{H}_1^7 = \varphi_9|_{T_h^7}; \quad \mathcal{H}_2^7 = \varphi_{11}|_{T_h^7}; \quad \mathcal{H}_3^7 = \varphi_{24}|_{T_h^7}.$$

N5

Expression for $\mathcal{H}^k_\alpha$, $\alpha = 1, 2, 3$:

$$\mathcal{H}^k_\alpha = c^k_\alpha + c^k_{x\,\alpha}\, x + c^k_{y\,\alpha}\, y,$$

$$\begin{pmatrix} 1 & x^k_1 & y^k_1 \\ 1 & x^k_2 & y^k_2 \\ 1 & x^k_3 & y^k_3 \end{pmatrix} \begin{pmatrix} c^k_\alpha \\ c^k_{x\,\alpha} \\ c^k_{y\,\alpha} \end{pmatrix} = \overbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}^{\alpha=1} \text{ or } \overbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}^{\alpha=2} \text{ or } \overbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}^{\alpha=3}$$

$$\Rightarrow \quad c^k_\alpha,\ c^k_{x\,\alpha},\ c^k_{y\,\alpha}\ , \quad \alpha = 1, 2, 3\ .$$

Implementation

Elemental Quantities

Elemental Matrices...

$$\tilde{A}_{h\,ij} = a(\varphi_i, \varphi_j) = \int_\Omega \frac{\partial \varphi_i}{\partial x}\frac{\partial \varphi_j}{\partial x} + \frac{\partial \varphi_i}{\partial y}\frac{\partial \varphi_j}{\partial y}\, dA$$

Element T_h^7 (say) contributes

$$\int_{T_h^7} \frac{\partial \varphi_{9,11,\text{ or }24}}{\partial x}\frac{\partial \varphi_{9,11,\text{ or }24}}{\partial x} + \frac{\partial \varphi_{9,11,\text{ or }24}}{\partial y}\frac{\partial \varphi_{9,11,\text{or }24}}{\partial y}\, dA\ .$$

Implementation

Elemental Quantities

...Elemental Matrices...

But

$$\int_{T_h^7} \frac{\partial \varphi_{9,11,\text{ or } 24}}{\partial x} \frac{\partial \varphi_{9,11,\text{ or } 24}}{\partial x} + \frac{\partial \varphi_{9,11,\text{ or } 24}}{\partial y} \frac{\partial \varphi_{9,11,\text{ or } 24}}{\partial y} \, dA$$

$$= \int_{T_h^7} \underbrace{\frac{\partial \mathcal{H}^7_{1,2,\text{ or } 3}}{\partial x}}_{\text{constant}} \underbrace{\frac{\partial \mathcal{H}^7_{1,2,\text{ or } 3}}{\partial x}}_{\text{constant}} + \underbrace{\frac{\partial \mathcal{H}^7_{1,2,\text{ or } 3}}{\partial y}}_{\text{constant}} \underbrace{\frac{\partial \mathcal{H}^7_{1,2,\text{ or } 3}}{\partial y}}_{\text{constant}} \, dA.$$

Define elemental matrices $\underline{A}^k \in \mathbb{R}^{3\times 3}$:

$$A^k_{\alpha\beta} = \int_{T_h^k} \frac{\partial \mathcal{H}^k_\alpha}{\partial x}\frac{\partial \mathcal{H}^k_\beta}{\partial x} + \frac{\partial \mathcal{H}^k_\alpha}{\partial y}\frac{\partial \mathcal{H}^k_\beta}{\partial y}\, dA$$

$$= \mathrm{Area}^k (c^k_{x\,\alpha}\, c^k_{x\,\beta} + c^k_{y\,\alpha}\, c^k_{y\,\beta}), \qquad 1 \le \alpha, \beta \le 3$$

since derivatives are all *constant* over T_h^k. **E1**

Implementation

Elemental Quantities

Elemental Loads...

$$\tilde{F}_{h\,i} = \ell(\varphi_i) = \int_\Omega f\,\varphi_i\,dA$$

Element T_h^7 (say) contributes

$$\int_{T_h^7} f\,\varphi_{9,11,\text{ or }24}\,dA$$

$$= \int_{T_h^7} f\,\mathcal{H}^7_{1,2,\text{ or }3}\,dA\ .$$

Implementation

Elemental Quantities

...Elemental Loads

Define elemental load vectors $\underline{F}^k \in \mathbb{R}^3$:

$$F_\alpha^k = \int_{T_h^k} f\,\mathcal{H}_\alpha^k\,dA, \qquad \alpha = 1, 2, 3\ ;$$

evaluation — approximation — of integral typically by numerical quadrature techniques.

Implementation

Assembly

The $\theta(k, \alpha)$ Array

Introduce local-to-global mapping

$$\theta(k,\alpha)\colon \underbrace{\{1,\ldots,K\}}_{\text{element}} \times \underbrace{\{1,2,3\}}_{\text{local node}} \to \underbrace{\{1,\ldots,\tilde{n}\}}_{\text{global node}}$$

such that

x_α^k (local node α in element k) $=$

$x_{\theta(k,\alpha)}$ (global node $\theta(k,\alpha)$).

Implementation

Assembly

Procedure for $\tilde{\underline{A}}_h$

To form $\underline{\widetilde{A}}_h$:

zero $\underline{\widetilde{A}}_h$;

$\{$for $k = 1, \ldots, K$

$\{$for $\alpha = 1, 2, 3$

$i = \theta(k, \alpha)$;

$\{$for $\beta = 1, 2, 3$

$j = \theta(k, \beta)$;

$\widetilde{A}_{h\,i\,j} = \widetilde{A}_{h\,i\,j} + A^k_{\alpha\beta}$; $\}$ $\}$ $\}$

E2

Implementation

Assembly

Procedure for $\underline{\tilde{F}}_h$

To form $\underline{\widetilde{F}}_h$:

zero $\underline{\widetilde{F}}_h$;

$\{$for $k = 1, \ldots, K$

$\{$for $\alpha = 1, 2, 3$

$i = \theta(k, \alpha)$;

$$\widetilde{F}_{h\,i} = \widetilde{F}_{h\,i} + F^k_\alpha \;; \} \; \}$$

Implementation

Boundary Conditions

Homogeneous Dirichlet...

Recall:

$$u_h \in X_h \qquad \boxed{u_h|_\Gamma = 0}$$

$$a(u_h, v) = \ell(v), \quad \forall\, v \in X_h \qquad \boxed{v|_\Gamma = 0\ ;}$$

$X_h = \text{span}\,\{\varphi_1, \ldots, \varphi_n\}$ *versus*

$$\tilde{X}_h = \text{span}\,\{\varphi_1, \ldots, \varphi_n,\ \varphi_{n+1}, \ldots, \varphi_{\tilde{n}}\}\ .$$

Implementation

Boundary Conditions

...Homogeneous Dirichlet...

Explicit Elimination

$\underline{X}_h \;\Rightarrow\; \varphi_{n+1}, \ldots, \varphi_{\tilde{n}}$ not admissible variations, so

REMOVE $Rn+1, \ldots, R\tilde{n}$ from $\underline{\tilde{A}}_h$;

$\tilde{u}_{h\,n+1}, \ldots, \tilde{u}_{h\,\tilde{n}} = 0$, so

REMOVE $Cn+1, \ldots, C\tilde{n}$ from $\underline{\tilde{A}}_h$.

Implementation

Boundary Conditions

...Homogeneous Dirichlet

Big Number Approach

Place $\frac{1}{\varepsilon}$ $(\varepsilon \ll 1)$ on entries $\tilde{A}_{h\,ii}$, $i = n+1, \ldots, \tilde{n}$.

Place $\mathbf{0}$ on entries $\tilde{F}_{h\,i}$, $i = n+1, \ldots, \tilde{n}$.

This replaces $Rn+1, \ldots, R\tilde{n}$ with
$u_{h\,n+1} \cong \cdots \cong u_{h\,\tilde{n}} \cong 0$ in an easy, symmetric way.

Implementation

Quadrature

How do we evaluate

$$F_{\alpha}^{k} = \int_{T_h^k} f(x)\, \mathcal{H}_{\alpha}^{k}(x)\, dA$$

for general f?

Implementation

Quadrature

Gauss Quadrature...

Approximate

$$F_\alpha^k = \int_{T_h^k} f(x)\, \mathcal{H}_\alpha^k \, dA$$

$$\approx \sum_{q=1}^{N_q} \rho_q^k \, f(z_q^k)\, \mathcal{H}_\alpha^k(z_q^k) \;;$$

ρ_q^k: quadrature weights,

z_q^k: quadrature points.

Implementation

Quadrature

...Gauss Quadrature

For example:

$$N_q = 1, \ \rho_1^k = \text{Area}^k, \ z_1^k = \frac{1}{3}(x_1^k + x_2^k + x_3^k)$$ N6

integrates exactly $\int_{T_h^k} g(x)\, dA$

for all $g \in \mathbb{P}_1(T_h^k)$;

higher order formulas tabulated.

Solution Methods for $\underline{A}_h \, \underline{u}_h = \underline{F}_h$

Overview

Topics

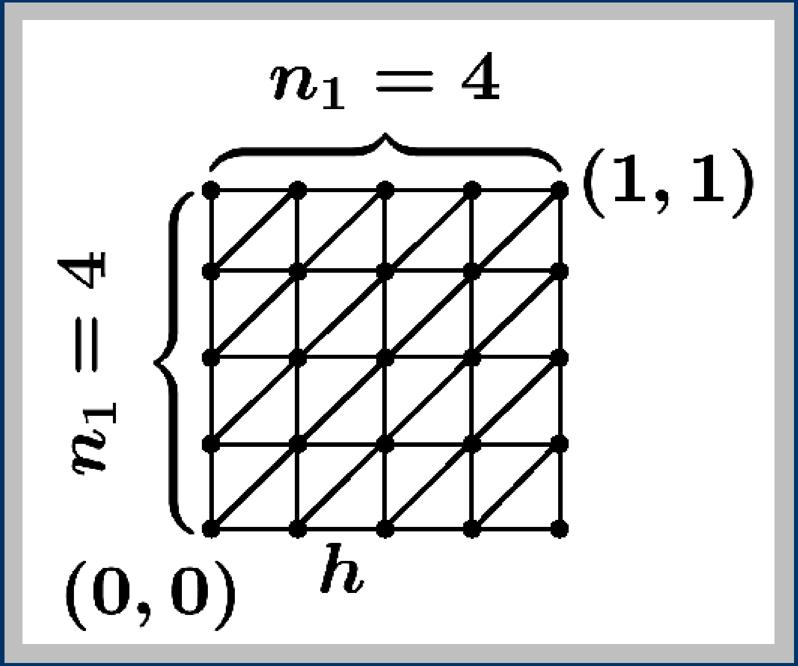
Direct Methods — Banded LU.

Iterative Methods — Conjugate Gradients:
algorithm and interpretation;
convergence rate and conditioning;
action of $\underline{A}_h$.

Solution Methods for $\underline{A}_h \, \underline{u}_h = \underline{F}_h$

Direct Methods — Banded LU

Uniform Mesh...



$$K = 2n_1^2$$

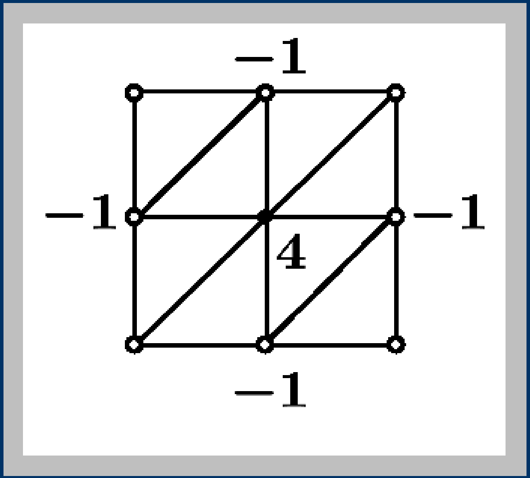
$$\tilde{n} = (n_1 + 1)^2$$

$$n = (n_1 - 1)^2$$

$$h = 1/n_1$$

Stencil

Nonzero entries of row i of $\underline{A}_h$:



Direct Methods — Banded LU

Operation Count and Storage

For "x–then–y" node numbering,

bandwidth $b = O(n_1)$.

LU: $O(n_1^2 \, n_1^2)$ operations; $O(n_1^2 \, n_1)$ storage.

Forward/Back Solves: $O(n_1^2 \, n_1)$ operations.

Solution Methods for $\underline{A}_h \, \underline{u}_h = \underline{F}_h$

Conjugate Gradient Iteration

Algorithm

N7

$\underline{u}_h = 0$ (say); $\underline{r}^0 = \underline{F}_h$;

For $k = 1, \ldots$:

$$\left.\begin{aligned} \beta^k &= (\underline{r}^{k-1})^T \, \underline{r}^{k-1} / (\underline{r}^{k-2})^T \, \underline{r}^{k-2} \\ \underline{p}^k &= \underline{r}^{k-1} + \beta^k \, \underline{p}^{k-1} \end{aligned}\right\} \underline{p}^1 = \underline{r}^0$$

$$\alpha^k = (\underline{r}^{k-1})^T \, \underline{r}^{k-1} / (\underline{p}^k)^T \, (\underline{A}_h \, \underline{p}^k)$$

$$\underline{u}_h^k = \underline{u}_h^{k-1} + \alpha^k \, \underline{p}^k$$

$$\underline{r}^k = \underline{r}^{k-1} - \alpha^k (\underline{A}_h \underline{p}^k) \, .$$

Solution Methods for $\underline{A}_h\,\underline{u}_h = \underline{F}_h$

Conjugate Gradient Iteration

Convergence Rate...

In general,

$$\frac{(\underline{u}_h - \underline{u}_h^k)^T \underline{A}_h (\underline{u}_h - \underline{u}_h^k)}{(\underline{u}_h)^T \underline{A}_h \underline{u}_h} \le 2 \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^k ,$$

$$\kappa(\underline{A}_h) = \frac{\lambda_{\max}(\underline{A}_h)}{\lambda_{\min}(\underline{A}_h)} .$$

Solution Methods for $\underline{A}_h \, \underline{u}_h = \underline{F}_h$

Conjugate Gradient Iteration

...Convergence Rate

For FEM $\underline{A}_h$:

$$\kappa(\underline{A}_h) \leq C h^{-2}$$

for quasi-uniform, regular meshes $\mathcal{T}_h$;

thus $n_{\text{iter}} \sim O(\frac{1}{h})$.

Solution Methods for $\underline{A}_h\,\underline{u}_h = \underline{F}_h$

Conjugate Gradient Iteration

Computational Effort

For uniform FEM mesh: $\frac{1}{h} = n_1$

$$n_{\text{iter}} \sim O(n_1)\ ;$$

work/iteration $\sim O(n_1^2)$; (Slide 44)

$\Rightarrow O(n_1^3)$ total operations, $O(n_1^2)$ storage.

Solution Methods for $\underline{A}_h\, \underline{u}_h = \underline{F}_h$

Conjugate Gradient Iteration

General Evaluation of $\underline{y} = \underline{A}_h\, \underline{p}$ ...

Given $\underline{\tilde{p}} \in \mathbb{R}^{\tilde{n}}$, $\tilde{p}_i = p_i, \; i = 1, \ldots, n$

$$\tilde{p}_i = 0, \; i = n+1, \ldots, \tilde{n}:$$

Evaluate $\underline{\tilde{y}} = \underline{\tilde{A}}_h\, \underline{\tilde{p}}$;

Set $y_i = \tilde{y}_i, \; i = 1, \ldots, n,$; $\tilde{y}_i = 0, \; i = n+1, \ldots, \tilde{n}$.

N8

Solution Methods for $\underline{A}_h \, \underline{u}_h = \underline{F}_h$

Conjugate Gradient Iteration

...General Evaluation of $\underline{y} = \underline{A}_h \, \underline{p}$

Evaluation of $\underline{\tilde{A}}_h \underline{\tilde{p}}$: $O(K)$ *operations*

zero $\underline{\tilde{y}}$; $\{$for $k = 1, \ldots, K$ (elements)

$\{$for $\alpha = 1, 2, 3$

$i = \theta(k, \alpha)$;

$\{$for $\beta = 1, 2, 3$

$j = \theta(k, \beta)$;

$$\widetilde{y}_i = \widetilde{y}_i + \boxed{A^k_{\alpha\beta}} \, \widetilde{p}_j \; ; \} \; \} \; \}$$