Numerical Methods for Partial Differential Equations (16.920J/2.097J/SMA5212)

Course Outline

- Overview of PDE's (1)
- Finite differences methods (6)
- Finite volume methods (3)
- Finite element methods (7)
- Boundary integral methods (6)
- Solution methods (3)

Total : 26 lectures

Assessment

Four Problem Sets/Mini-projects:

Finite Differences	25 %
Hyperbolic Equations	20 %
Finite Elements	25 %
Boundary Integral Methods	20 %

Class Interaction	10 %

Partial Differential Equations: An Overview

Lecture 1

Model Equation

Convection-Diffusion

$$\boxed{\frac{\partial u}{\partial t} + U \cdot \nabla u = \kappa \nabla^2 u + f}$$

N1

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right), \qquad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

U, $\kappa > 0$, f, given functions of (x, y)

Scalar, Linear, Parabolic equation

N2

Model Equation

Convection-Diffusion

Applications

$$\frac{\partial u}{\partial t} + U \cdot \nabla u = \kappa \nabla^2 u + f$$

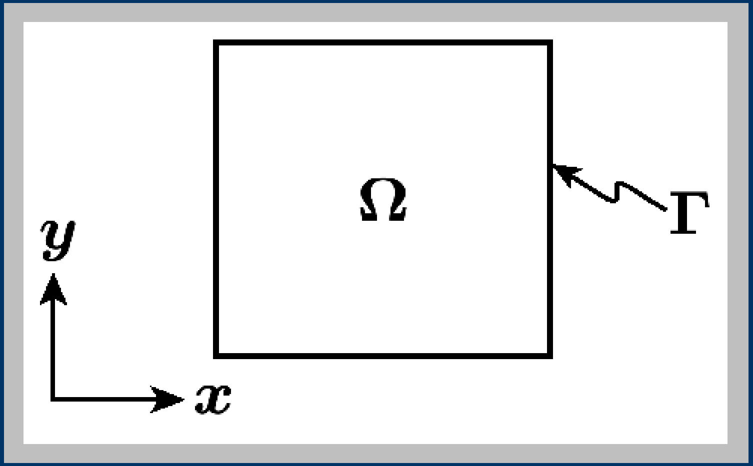
If u is . . .

- Temperature $\rightarrow$ **Heat Transfer**
- Pollutant Concentration $\rightarrow$ **Coastal Engineering**
- Probability Distribution $\rightarrow$ **Statistical Mechanics**
- Price of an Option $\rightarrow$ **Financial Engineering**
- . . .

Limiting Cases

Poisson Equation

$$-\kappa\nabla^2 u = f \quad \text{in } \Omega$$



Convection-Diffusion

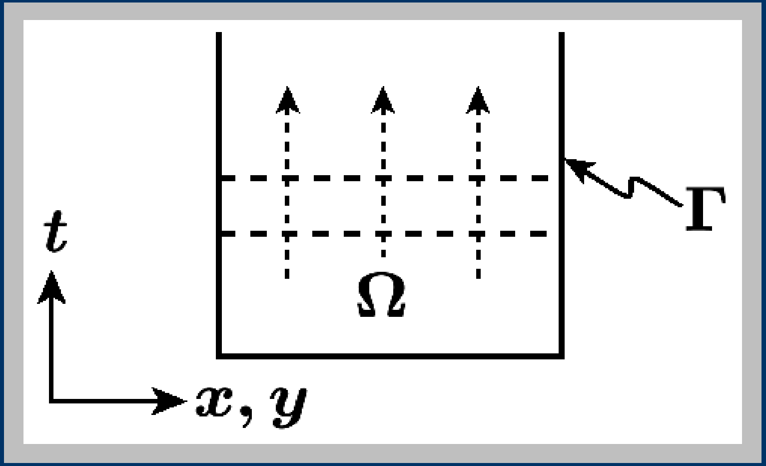
$$U \cdot \nabla u = \kappa\nabla^2 u \quad \text{in } \Omega$$

- "Smooth" solutions
- The domain of dependence of $u(x, y)$ is Ω

Limiting Cases

Heat Equation

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + f \quad \text{in } \Omega$$

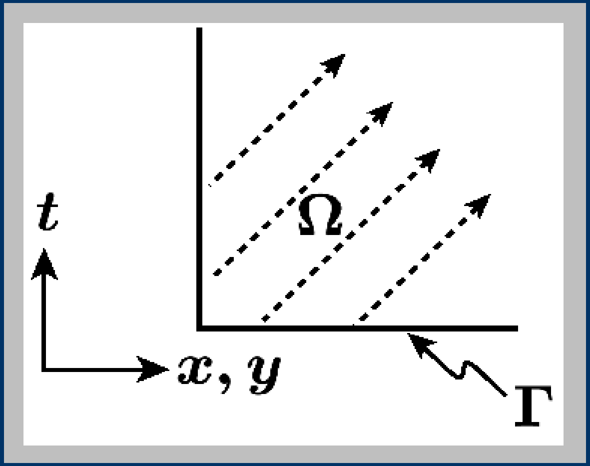


- "Smooth" solutions
- The domain of dependence of $u(x, y, T)$ is $(x, y, t < T)$

Limiting Cases

Wave Equation (First order)

$$\frac{\partial u}{\partial t} + U \cdot \nabla u = f \quad \text{in } \Omega$$

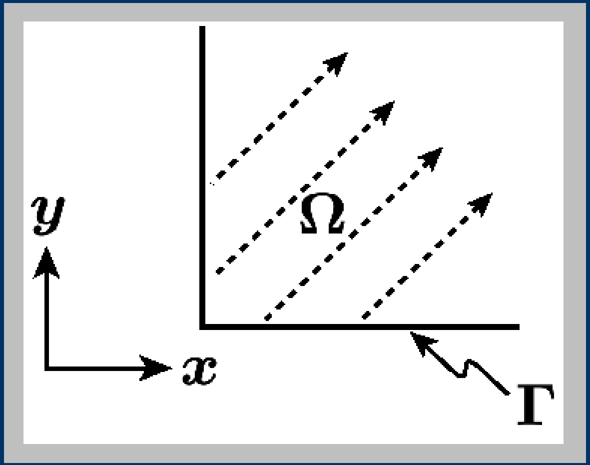


- Non-smooth solutions
- Characteristics : $\frac{dx_c}{dt} = U(x_c(t))$
- The domain of dependence of $u(x, T)$ is $(x_c(t), t < T)$

Limiting Cases

Convection Equation

$$U \cdot \nabla u = f \quad \text{in } \Omega$$

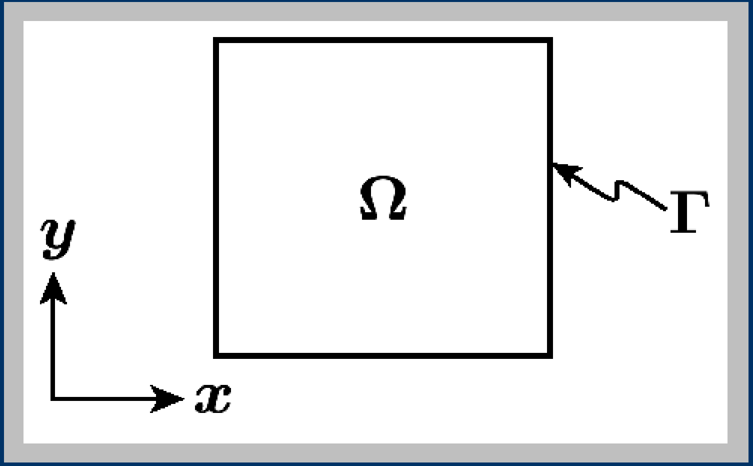


- Non-smooth solutions
- Characteristics are streamlines of U, e.g. $\frac{dx_c}{ds} = U$
- The domain of dependence of $u(x)$ is $(x_c(s), s < 0)$

Eigenvalue Problem

Limiting Cases

Find non-trivial pairs (u, λ)



$$\kappa \nabla^2 u + \lambda u = 0 \quad \text{in} \quad \Omega$$

with **homogeneous** conditions on Γ

- Non-linear
- "Closely" related to other problems

Limiting Cases

One Spatial Variable

Unknown	Equation
$u(x)$:	$-u_{xx} = f$
$u(x)$:	$Uu_x = \kappa u_{xx}$
$u(x,t)$:	$u_t = \kappa u_{xx}$
$u(x,t)$:	$u_t + Uu_x = 0$
$(u(x), \lambda)$:	$u_{xx} + \lambda u = 0$

Fourier Analysis

Definition

Let $g(x)$ be an "arbitrary" periodic real function with period 2π

N3

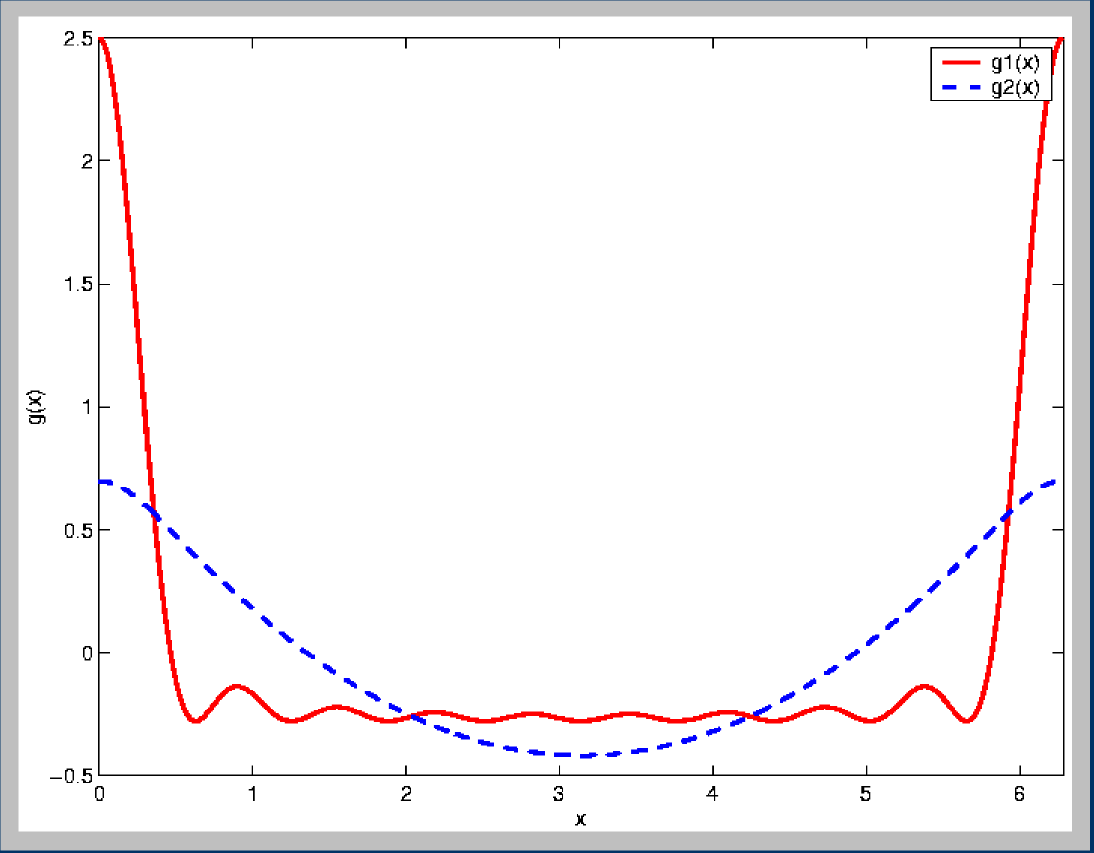
$$g(x) = \sum_{k=-\infty}^{\infty} g_k \, e^{ikx} \quad (k \text{ integer}) \ .$$

$$\int_0^{2\pi} e^{ikx} e^{-ik'x} \, dx = 2\pi \, \delta_{kk'} \quad \text{(orthogonality)}$$

$$g_k = \frac{1}{2\pi} \int_0^{2\pi} g(x) \, e^{-ikx} \, dx$$

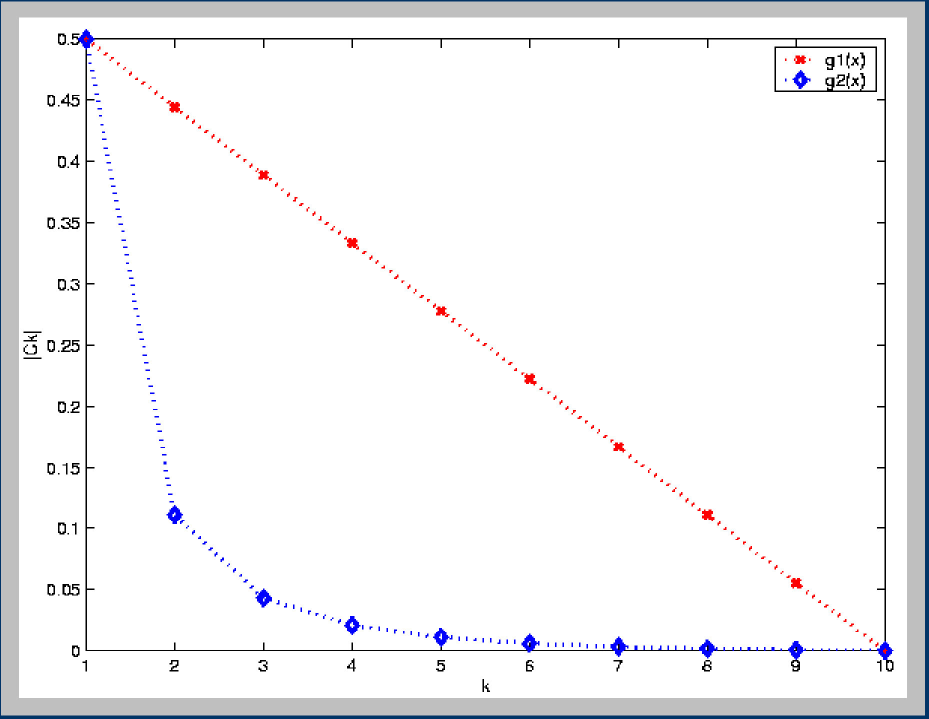
Fourier Analysis

Example



Fourier Analysis

Example



Rate at which $|g_k| \to 0$ for $|k|$ large determines **smoothness**

Fourier Analysis

Differentiation

$$u(x) = \sum_{k=-\infty}^{\infty} u_k \, e^{ikx} \quad \text{or} \quad u(x,t) = \sum_{k=-\infty}^{\infty} u_k(t) \, e^{ikx}$$

$$\frac{\partial^n u}{\partial x^n} = \sum_{k=-\infty}^{\infty} (ik)^n u_k e^{ikx} \qquad \frac{\partial u}{\partial t} = \sum_{k=-\infty}^{\infty} \frac{du_k}{dt} e^{ikx}$$

$$n = 2m \quad \rightarrow \quad (ik)^n = (-1)^m \, k^{2m} \qquad \text{(real)}$$

$$n = 2m-1 \quad \rightarrow \quad (ik)^n = -i(-1)^m \, k^{2m-1} \qquad \text{(imaginary)}$$

Fourier Analysis

Poisson Equation

$$-u_{xx} = f \qquad x \in (0, 2\pi)$$

with

$$u(0) = u(2\pi),$$

$$u_x(0) = u_x(2\pi),$$

and

$$\int_0^{2\pi} u\, dx = 0, \qquad \int_0^{2\pi} f\, dx = 0$$

N4

Poisson Equation

Fourier Analysis

$$u = \sum_{k=-\infty}^{\infty} u_k \, e^{ikx} \, , \quad f = \sum_{k=-\infty}^{\infty} f_k \, e^{ikx} \qquad (f_0 = 0)$$

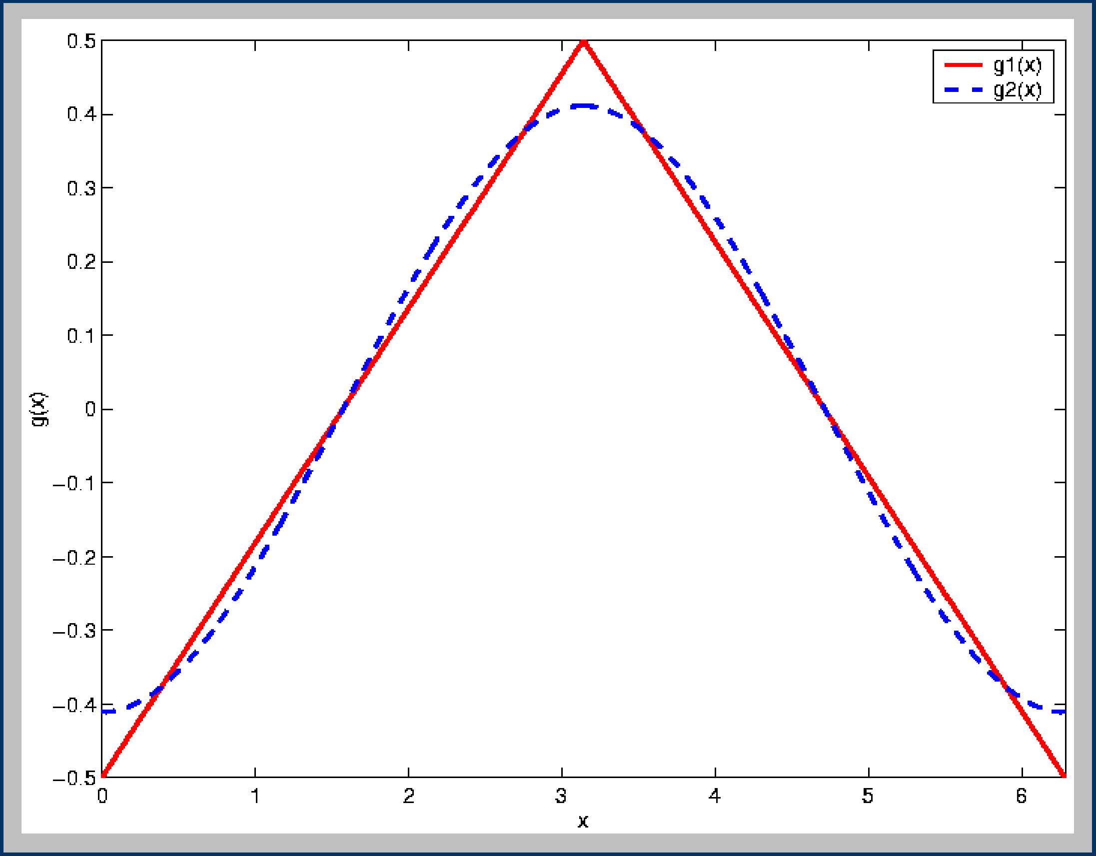
$$-u_{xx} = \sum_{k=-\infty}^{\infty} k^2 \, u_k \, e^{ikx} \quad \rightarrow \quad \boxed{u_k = \frac{f_k}{k^2}} \quad (u_0 = 0)$$

$\Rightarrow$ – the solution u is **smoother** than f

Fourier Analysis

Poisson Equation

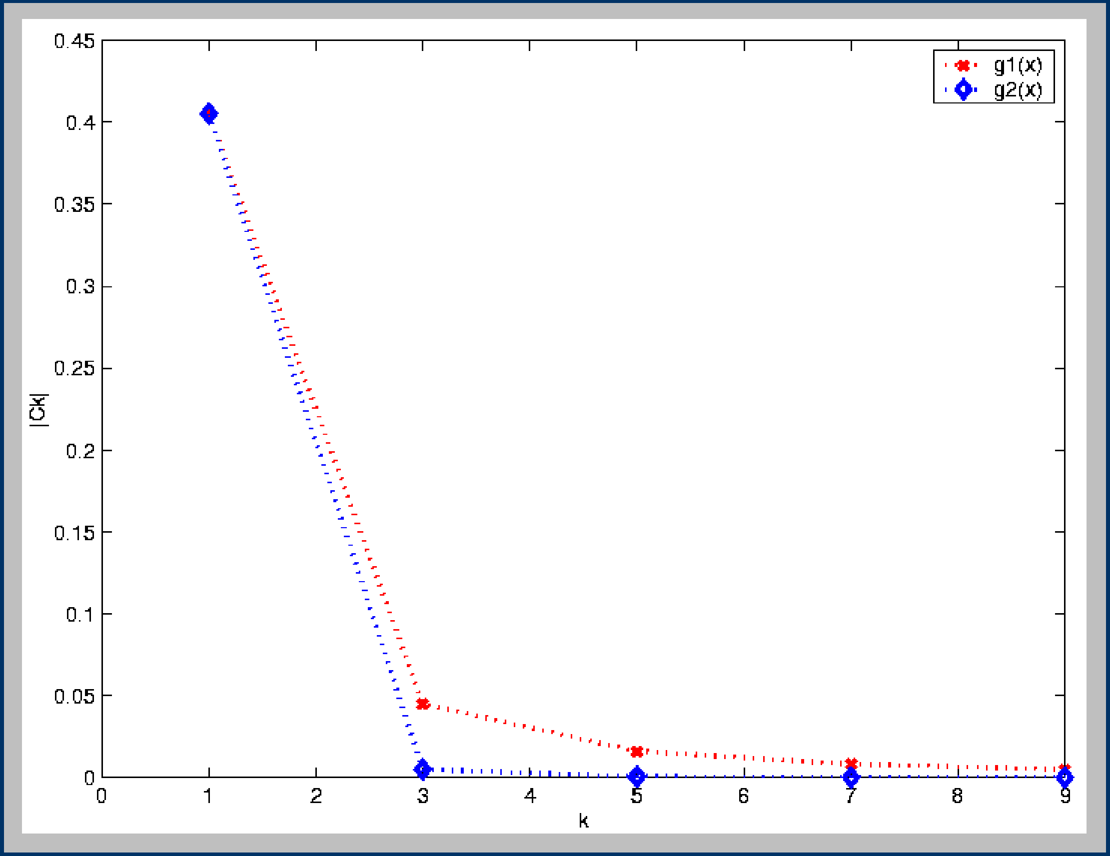
Example...



Fourier Analysis

Poisson Equation

...Example



Fourier Analysis

Heat Equation

$$u_t = \kappa u_{xx} \qquad x \in (0, 2\pi)$$

with

$$u(0,t) = u(2\pi, t),$$

$$u_x(0,t) = u_x(2\pi, t),$$

$$u(x,0) = u^0(x) = \sum_{k=-\infty}^{\infty} u_k^0 \, e^{ikx}$$

Fourier Analysis

Heat Equation

$$u = \sum_{k=-\infty}^{\infty} u_k(t)\, e^{ikx}$$

$$u_t = \sum_{k=-\infty}^{\infty} \frac{du_k}{dt}\, e^{ikx}\,, \qquad u_{xx} = \sum_{k=-\infty}^{\infty} -k^2\, u_k\, e^{ikx}$$

$$\frac{du_k}{dt} = -\kappa k^2\, u_k$$

Fourier Analysis

Heat Equation

$$\frac{du_k}{dt} = -\kappa k^2 u_k \,, \quad u_k(t=0) = u_k^0 \,, \;\Rightarrow\; u_k(t) = u_k^0 e^{-\kappa k}$$

$$\boxed{u(x,t) = \sum_{k=-\infty}^{\infty} u_k^0 \, e^{-\kappa k^2 t} \, e^{ikx}}$$

$\Rightarrow$

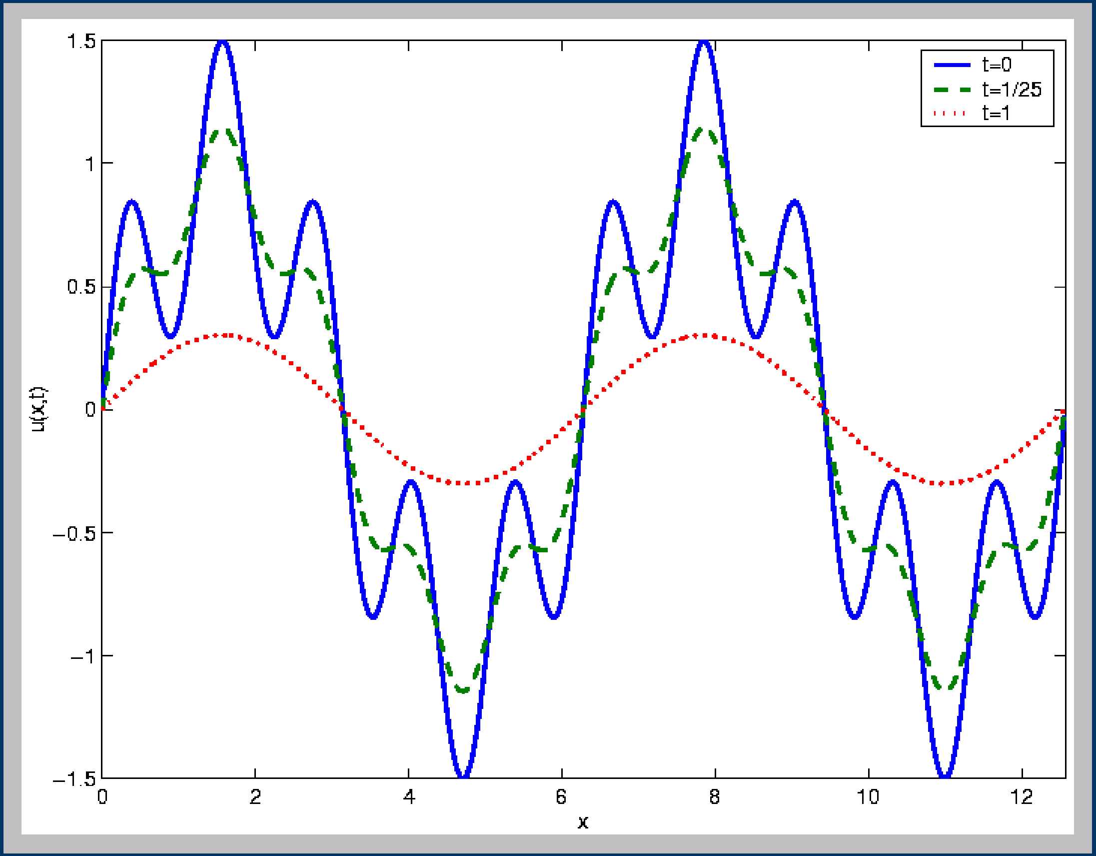
– exponential decay of initial condition (**dissipation**)

– higher decay for "higher modes" (larger k) $\equiv$ **smoothness**

Fourier Analysis

Heat Equation

Example



Fourier Analysis

Wave Equation

$$u_t + U u_x = 0 \qquad x \in (0, 2\pi)$$

with

$$u(0,t) = u(2\pi, t),$$

$$u(x,0) = u^0(x) = \sum_{k=-\infty}^{\infty} u_k^0 \, e^{ikx}$$

Fourier Analysis

Wave Equation

$$u = \sum_{k=-\infty}^{\infty} u_k(t)\, e^{ikx}$$

$$u_t = \sum_{k=-\infty}^{\infty} \frac{du_k}{dt}\, e^{ikx}\,, \qquad u_x = \sum_{k=-\infty}^{\infty} iku_k\, e^{ikx}$$

$$\frac{du_k}{dt} = -iUk\, u_k$$

Fourier Analysis

Wave Equation

$$\frac{du_k}{dt} = -iUku_k\,, \quad u_k(0) = u_k^0 \quad \Rightarrow \quad u_k(t) = u_k^0\, e^{-iUkt}$$

$$u(x,t) = \sum_{k=-\infty}^{\infty} u_k^0\, e^{-iUkt}\, e^{ikx} = \sum_{k=-\infty}^{\infty} u_k^0\, e^{ik(x-Ut)} = u^0(x - Ut)$$

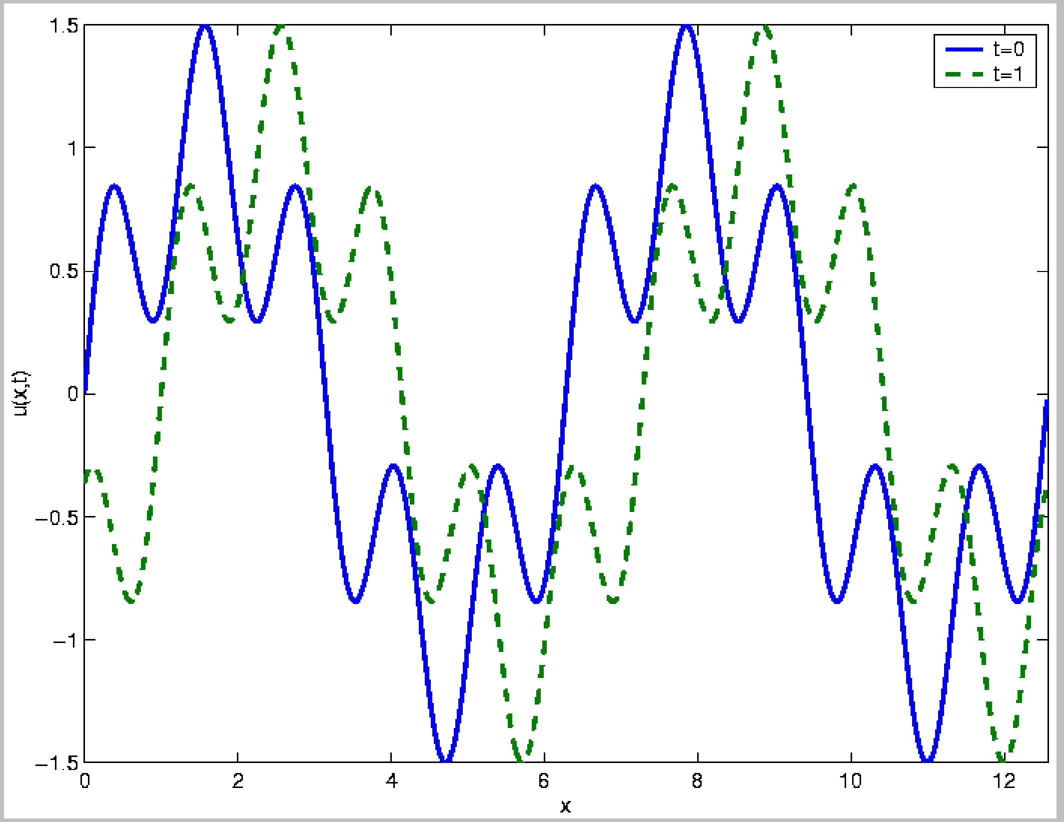
$\Rightarrow$

- no decay, **propagation** with wave speed $c = U$
- no **dispersion** (c constant) $\equiv$ invariant shape

Fourier Analysis

Wave Equation

Example



Fourier Analysis

General Operator

$$u_t = \frac{\partial^n u}{\partial x^n} \qquad x \in (0, 2\pi)$$

with

$$\begin{aligned} u(0,t) &= u(2\pi,t), \\ u_x(0,t) &= u_x(2\pi,t), \\ &\vdots \\ u_x^{(n-1)}(0,t) &= u_x^{(n-1)}(2\pi,t), \\ u(x,0) &= u^0(x) \end{aligned}$$

Fourier Analysis

General Operator

$$u = \sum_{k=-\infty}^{\infty} u_k(t)\, e^{ikx}$$

$$u_t = \sum_{k=-\infty}^{\infty} \frac{du_k}{dt}\, e^{ikx}\,, \qquad u_x^{(n)} = \sum_{k=-\infty}^{\infty} (ik)^n u_k\, e^{ikx}$$

$$\boxed{\frac{du_k}{dt} = \sigma\, u_k}$$

$$\sigma = (ik)^n$$

Fourier Analysis

General Operator

n	σ	Feature
1	ik	**Propagation**, $c = -\sigma/ik = -1$ (no Dispersion)
2	$-k^2$	**Decay**
3	$-ik^3$	**Propagation**, $c = +k^2$ (and **Dispersion**)
4	k^4	**Growth** ($-u_{xxxx}$ much faster Decay than u_{xx})
.	.	

N5

Fourier Analysis

Eigenvalue Problem

$$\boxed{u_{xx} + \lambda u = 0 \qquad x \in (0, 2\pi)}$$

with

$$u(0) = u(2\pi),$$

$$u_x(0) = u_x(2\pi)$$

Need to determine non-trivial pairs $(u^n(x), \lambda^n)$

Fourier Analysis

Eigenvalue Problem

It can be easily verified that the eigenvalues are:

$$\lambda^n = n^2, \quad \text{for} \quad n = 1, 2, \ldots$$

The eigenvectors associated with λ^n are:

$$u_1^n(x) = e^{inx}, \; u_2^n(x) = e^{-inx}, \quad \text{for} \quad n = 1, 2, \ldots$$

Eigenmodes $\equiv$ Fourier modes

Eigenvalue Expansions

Formal Extension

$$\frac{\partial u}{\partial t} = \mathcal{L}u$$

with homogeneous boundary conditions

N6

$$u(x, y, t) = \sum_{n=0}^{\infty} a_n(t) u^n(x, y)$$

(u^n, λ^n) solution of $\mathcal{L}u - \lambda u = 0$

Eigenvalue Expansions

Formal Extension

$$\mathcal{L}u = \sum_{n=0}^{\infty} \lambda^n a_n u^n, \qquad \frac{\partial u}{\partial t} = \sum_{n=0}^{\infty} \frac{da_n}{dt} u^n$$

$$\frac{da_n}{dt} = \lambda^n a_n \quad \Rightarrow \quad a_n(t) = a_n^0 \, e^{\lambda^n t}$$

$$\boxed{u(x,y,t) = \sum_{n=0}^{\infty} a_n^0 \, e^{\lambda^n t} \, u^n(x,y)}$$

Eigenvalue Expansions

Formal Extension

Eigenvalues determine temporal evolution of the associated time-dependent problem.

Higher λ

$\Updownarrow$

Higher **decay/frequency**

$\Updownarrow$

More Oscillations