Midterm Examination

15.084J/6.252J: Nonlinear Programming

March 19, 1998

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This is a closed book exam.

Please write your solutions in the white exam books provided. Be clear. Your grade will depend in part on the clarity of your answers.

The point count for each question is given next to the question.

Question 1: (10 points)

In the lectures, we have spent considerable time developing and discussing two methods for unconstrained optimization: the method of steepest descent, and Newton's method.

(a) List several advantages that the method of steepest descent has over Newton's method. Please give only a *brief* explanation of each advantage.

(b) List several advantages that Newton's method has over the method of steepest descent. Please give only a *brief* explanation of each advantage.

Question 2: (15 points)

Suppose that $f(x)$ is a differentiable convex function on $\Re^n$. Prove that for any $x, y \in \Re^n$, that

$$(\nabla f(x) - \nabla f(y))^T (x - y) \geq 0.$$

Question 3: (15 points)

Consider the problem (P):

$$(P): \quad \text{minimize} \qquad\qquad -x_1 - x_2$$

$$\text{s.t.} \qquad -x_1^3 + 6x_1^2 - 9x_1 + x_2 - 10 \leq 0$$

$$-x_2 + 14 \leq 0$$

$$x_1 - 5 \leq 0.$$

(a) Does the point $(x_1, x_2) = (1, 14)$ satisfy the Karush-Kuhn-Tucker and/or the Fritz John necessary conditions for a local minimum? Why or why not?

(b) Consider the general program $(\bar{P})$:

$$(\bar{P}): \quad \text{minimize} \quad f(x)$$

$$\text{s.t.} \qquad g_i(x) \leq 0 , \quad i = 1, \ldots, m$$

$$x \in \Re^n.$$

Suppose that $\bar{x}$ is the unique point satisfying the Karush-Kuhn-Tucker conditions for $(\bar{P})$. Is $\bar{x}$ the optimal solution to $(\bar{P})$? Support your answer.

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Question 4: (15 points)

Let $\bar{x}$ satisfy $A\bar{x} = b$ in the problem:

$$\text{minimize} \quad f(x)$$

$$\text{s.t.} \quad Ax = b$$

$$x \in \Re^n,$$

where $f(x)$ is a continuously differentiable function. Let $\nabla f(\bar{x})$ denote the gradient of $f(x)$ at $x = \bar{x}$.

(a) Write down the direction finding problem for the projected steepest descent algorithm at $x = \bar{x}$, in the case that the matrix Q is $Q = I$.

(b) Consider the following Euclidean projection problem:

$$Proj(\bar{x}): \quad \underset{d}{\text{minimize}} \quad \tfrac{1}{2}\left(\|-\nabla f(\bar{x}) - d\|\right)^2$$

$$\text{s.t.} \quad Ad = 0.$$

What are the Karush-Kuhn-Tucker conditions for $Proj(\bar{x})$? Are these conditions necessary and sufficient? Why or why not?

(c) Let $\bar{d}$ denote the projected steepest descent direction and let $\tilde{d}$ denote the solution to $Proj(\bar{x})$. Show that these two directions are the same (up to a scalar multiple).

Question 5: (15 points)

Consider the steepest descent algorithm applied to the problem:

$$(P): \quad \text{minimize}_x \quad \tfrac{1}{2}x^T Q x + c^T x$$

where Q is a symmetric and positive definite matrix.

(a) Consider a linear transformation of the variables $y = Mx$ where M is an invertible $n \times n$ matrix. Rewrite (P) as a minimization problem over the variables y. Call this transformed problem (P_M). What is the marix of the quadratic objective function of (P_M) ?

(b) In running the steepest descent algorithm, the convergence rate is sensitive to the eigenvalue structure of the matrix Q. Suppose that you know this eigenvalue structure, that is, you know an orthonormal matrix R and a diagonal matrix D for which

$$Q = RDR^T.$$

Devise a matrix M based on R and D for which the transformed problem (P_M) has a real "nice" eigenvalue structure, and for which the steepest descent algorithm will converge in just one step.

Question 6: (15 points)

Let X be an open set in $\Re^n$. Suppose that that $h(x)$ is a nonnegative convex function for all $x \in X$, and that $g(x)$ is strictly positive concave function for all $x \in X$. Show that the function:

$$f(x) = \frac{h(x)}{g(x)}$$

is a quasiconvex function on X.

HINT: Look at the level sets of the function $f(x)$.