11 Cardinality Rules

11.1 Counting One Thing by Counting Another

How do you count the number of people in a crowded room? You could count heads, since for each person there is exactly one head. Alternatively, you could count ears and divide by two. Of course, you might have to adjust the calculation if someone lost an ear in a pirate raid or someone was born with three ears. The point here is that you can often *count one thing by counting another*, though some fudge factors may be required. This is a central theme of counting, from the easiest problems to the hardest.

In more formal terms, every counting problem comes down to determining the size of some set. The *size* or *cardinality* of a finite set S is the number of elements in S and it is denoted by $|S|$. In these terms, we're claiming that we can often find the size of one set by finding the size of a related set. We've already seen a general statement of this idea in the Mapping Rule of Theorem 7.2.1. Of particular interest here is part 3 of Theorem 7.2.1, where we state that if there is a bijection between two sets, then the sets have the same size. This important fact is commonly known as the *Bijection Rule*.

11.1.1 The Bijection Rule

Rule 11.1.1 (Bijection Rule)**.** *If there is a bijection $f : A \to B$ between A and B, then $|A| = |B|$.*

The Bijection Rule acts as a magnifier of counting ability; if you figure out the size of one set, then you can immediately determine the sizes of many other sets via bijections. For example, consider the two sets mentioned at the beginning of Part III:

A = all ways to select a dozen doughnuts when five varieties are available

B = all 16-bit sequences with exactly 4 ones

Let's consider a particular element of set A:

$$\underbrace{0\;0}_{\text{chocolate}}\qquad\underbrace{\qquad}_{\text{lemon-filled}}\qquad\underbrace{0\;0\;0\;0\;0\;0}_{\text{sugar}}\qquad\underbrace{0\;0}_{\text{glazed}}\qquad\underbrace{0\;0}_{\text{plain}}$$

We've depicted each doughnut with a 0 and left a gap between the different varieties. Thus, the selection above contains two chocolate doughnuts, no lemon-filled, six sugar, two glazed, and two plain. Now let's put a 1 into each of the four gaps:

$$\underbrace{0\,0}_{\text{chocolate}}\quad 1 \quad \underbrace{\qquad}_{\text{lemon-filled}}\quad 1 \quad \underbrace{0\,0\,0\,0\,0\,0}_{\text{sugar}}\quad 1 \quad \underbrace{0\,0}_{\text{glazed}}\quad 1 \quad \underbrace{0\,0}_{\text{plain}}$$

We've just formed a 16-bit number with exactly 4 ones—an element of B!

This example suggests a bijection from set A to set B: map a dozen doughnuts consisting of:

$$c \text{ chocolate, } l \text{ lemon-filled, } s \text{ sugar, } g \text{ glazed, and } p \text{ plain}$$

to the sequence:

$$\underbrace{0\ldots0}_{c}\quad 1 \quad \underbrace{0\ldots0}_{l}\quad 1 \quad \underbrace{0\ldots0}_{s}\quad 1 \quad \underbrace{0\ldots0}_{g}\quad 1 \quad \underbrace{0\ldots0}_{p}$$

The resulting sequence always has 16 bits and exactly 4 ones, and thus is an element of B. Moreover, the mapping is a bijection; every such bit sequence is mapped to by exactly one order of a dozen doughnuts. Therefore, $|A| = |B|$ by the Bijection Rule!

This example demonstrates the magnifying power of the bijection rule. We managed to prove that two very different sets are actually the same size—even though we don't know exactly how big either one is. But as soon as we figure out the size of one set, we'll immediately know the size of the other.

This particular bijection might seem frighteningly ingenious if you've not seen it before. But you'll use essentially this same argument over and over, and soon you'll consider it routine.

11.2 Counting Sequences

The Bijection Rule lets us count one thing by counting another. This suggests a general strategy: get really good at counting just a *few* things and then use bijections to count *everything else*. This is the strategy we'll follow. In particular, we'll get really good at counting *sequences*. When we want to determine the size of some other set T, we'll find a bijection from T to a set of sequences S. Then we'll use our super-ninja sequence-counting skills to determine $|S|$, which immediately gives us $|T|$. We'll need to hone this idea somewhat as we go along, but that's pretty much the plan!

11.2.1 The Product Rule

The *Product Rule* gives the size of a product of sets. Recall that if $P_1, P_2, \ldots, P_n$ are sets, then

$$P_1 \times P_2 \times \ldots \times P_n$$

is the set of all sequences whose first term is drawn from P_1, second term is drawn from P_2 and so forth.

Rule 11.2.1 (Product Rule)**.** *If $P_1, P_2, \ldots P_n$ are sets, then:*

$$|P_1 \times P_2 \times \ldots \times P_n| = |P_1| \cdot |P_2| \cdots |P_n| \leftarrow$$

For example, suppose a *daily diet* consists of a breakfast selected from set B, a lunch from set L, and a dinner from set D where:

$$\begin{aligned} B &= \{\text{pancakes, bacon and eggs, bagel, Doritos}\} \leftarrow \\ L &= \{\text{burger and fries, garden salad, Doritos}\} \leftarrow \\ D &= \{\text{macaroni, pizza, frozen burrito, pasta, Doritos}\} \leftarrow \end{aligned}$$

Then $B \times L \times D$ is the set of all possible daily diets. Here are some sample elements:

(pancakes, burger and fries, pizza)
(bacon and eggs, garden salad, pasta)
(Doritos, Doritos, frozen burrito)

The Product Rule tells us how many different daily diets are possible:

$$\begin{aligned} |B \times L \times D| &= |B| \cdot |L| \cdot |D| \leftarrow \\ &= 4 \cdot 3 \cdot 5 \\ &= 60. \end{aligned}$$

11.2.2 Subsets of an *n*-element Set

How many different subsets of an n-element set X are there? For example, the set $X = \{x_1, x_2, x_3\}$ has eight different subsets:

$$\begin{array}{cccc} \emptyset & \{x_1\} & \{x_2\} & \{x_1, x_2\} \\ \{x_3\} & \{x_1, x_3\} & \{x_2, x_3\} & \{x_1, x_2, x_3\}. \end{array}$$

There is a natural bijection from subsets of X to n-bit sequences. Let $x_1, x_2, \ldots, x_n$ be the elements of X. Then a particular subset of X maps to the sequence $(b_1, \ldots, b_n)$

where $b_i = 1$ if and only if x_i is in that subset. For example, if $n = 10$, then the subset $\{x_2, x_3, x_5, x_7, x_{10}\}$ maps to a 10-bit sequence as follows:

$$\begin{array}{rccccccccccccc} \text{subset:} & \{ & & x_2, & x_3, & & x_5, & & x_7, & & & x_{10} & \} \\ \text{sequence:} & (& 0, & 1, & 1, & 0, & 1, & 0, & 1, & 0, & 0, & 1 &) \end{array}$$

We just used a bijection to transform the original problem into a question about sequences—*exactly according to plan!* Now if we answer the sequence question, then we've solved our original problem as well.

But how many different n-bit sequences are there? For example, there are 8 different 3-bit sequences:

$$\begin{array}{cccc} (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) \\ (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) \end{array}$$

Well, we can write the set of all n-bit sequences as a product of sets:

$$\underbrace{\{0,1\} \times \{0,1\} \times \ldots \times \{0,1\}}_{n \text{ terms}} = \{0,1\}^n$$

Then Product Rule gives the answer:

$$\begin{aligned} |\{0,1\}^n| &= |\{0,1\}|^n \\ &= 2^n \end{aligned}$$

This means that the number of subsets of an n-element set X is also 2^n. We'll put this answer to use shortly.

11.2.3 The Sum Rule

Linus allocates his big sister Lucy a quota of 20 crabby days, 40 irritable days, and 60 generally surly days. On how many days can Lucy be out-of-sorts one way or another? Let set C be her crabby days, I be her irritable days, and S be the generally surly. In these terms, the answer to the question is $|C \cup I \cup S|$. Now assuming that she is permitted at most one bad quality each day, the size of this union of sets is given by the *Sum Rule*:

Rule 11.2.2 (Sum Rule)**.** *If* $A_1, A_2, \ldots, A_n$ *are* disjoint *sets, then:*

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|$$

Thus, according to Linus' budget, Lucy can be out-of-sorts for:

$$\begin{aligned} |C \cup I \cup S| &= |C| + |I| + |S| \\ &= 20 + 40 + 60 \\ &= 120 \text{ days} \end{aligned}$$

Notice that the Sum Rule holds only for a union of *disjoint* sets. Finding the size of a union of intersecting sets is a more complicated problem that we'll take up later.

11.2.4 Counting Passwords

Few counting problems can be solved with a single rule. More often, a solution is a flurry of sums, products, bijections, and other methods. For example, the sum and product rules together are useful for solving problems involving passwords, telephone numbers, and license plates. For example, on a certain computer system, a valid password is a sequence of between six and eight symbols. The first symbol must be a letter (which can be lowercase or uppercase), and the remaining symbols must be either letters or digits. How many different passwords are possible?

Let's define two sets, corresponding to valid symbols in the first and subsequent positions in the password.

$$\begin{aligned} F &= \{a, b, \ldots, z, A, B, \ldots, Z\} \\ S &= \{a, b, \ldots, z, A, B, \ldots, Z, 0, 1, \ldots, 9\} \end{aligned}$$

In these terms, the set of all possible passwords is:[1]

$$(F \times S^5) \cup (F \times S^6) \cup (F \times S^7)$$

Thus, the length-six passwords are in the set $F \times S^5$, the length-seven passwords are in $F \times S^6$, and the length-eight passwords are in $F \times S^7$. Since these sets are disjoint, we can apply the Sum Rule and count the total number of possible passwords as follows:

$$\begin{aligned} &|(F \times S^5) \cup (F \times S^6) \cup (F \times S^7)| \\ &\quad = |F \times S^5| + |F \times S^6| + |F \times S^7| && \text{Sum Rule} \\ &\quad = |F| \cdot |S|^5 + |F| \cdot |S|^6 + |F| \cdot |S|^7 && \text{Product Rule} \\ &\quad = 52 \cdot 62^5 + 52 \cdot 62^6 + 52 \cdot 62^7 \\ &\quad \approx 1.8 \cdot 10^{14} \text{ different passwords.} \end{aligned}$$

11.3 The Generalized Product Rule

We realize everyone has been working pretty hard this term, and we're considering awarding some prizes for *truly exceptional* coursework. Here are some possible

[1] The notation S^5 means $S \times S \times S \times S \times S$.

categories:

Best Administrative Critique We asserted that the quiz was closed-book. On the cover page, one strong candidate for this award wrote, "There is no book."

Awkward Question Award "Okay, the left sock, right sock, and pants are in an antichain, but how—even with assistance—could I put on all three at once?"

Best Collaboration Statement Inspired by a student who wrote "I worked alone" on Quiz 1.

In how many ways can, say, three different prizes be awarded to n people? This is easy to answer using our strategy of translating the problem about awards into a problem about sequences. Let P be the set of n people taking the course. Then there is a bijection from ways of awarding the three prizes to the set $P^3 ::= P \times P \times P$. In particular, the assignment:

"person x wins prize #1, y wins prize #2, and z wins prize #3"

maps to the sequence (x, y, z). By the Product Rule, we have $|P^3| = |P|^3 = n^3$, so there are n^3 ways to award the prizes to a class of n people.

But what if the three prizes must be awarded to *different* students? As before, we could map the assignment

"person x wins prize #1, y wins prize #2, and z wins prize #3"

to the triple $(x, y, z) \in P^3$. But this function is *no longer a bijection*. For example, no valid assignment maps to the triple (Dave, Dave, Becky) because Dave is not allowed to receive two awards. However, there *is* a bijection from prize assignments to the set:

$$S = \{(x, y, z) \in P^3 \mid x, y, \text{ and } z \text{ are different people}\}$$

This reduces the original problem to a problem of counting sequences. Unfortunately, the Product Rule is of no help in counting sequences of this type because the entries depend on one another; in particular, they must all be different. However, a slightly sharper tool does the trick.

Rule 11.3.1 (Generalized Product Rule)**.** *Let S be a set of length-k sequences. If there are:*

- n_1 *possible first entries,*
- n_2 *possible second entries for each first entry,*

- *n_3 possible third entries for each combination of first and second entries, etc.*

then:

$$|S| = n_1 \cdot n_2 \cdot n_3 \cdots n_k$$

In the awards example, S consists of sequences (x, y, z). There are n ways to choose x, the recipient of prize #1. For each of these, there are $n-1$ ways to choose y, the recipient of prize #2, since everyone except for person x is eligible. For each combination of x and y, there are $n-2$ ways to choose z, the recipient of prize #3, because everyone except for x and y is eligible. Thus, according to the Generalized Product Rule, there are

$$|S| = n \cdot (n-1) \cdot (n-2)$$

ways to award the 3 prizes to different people.

11.3.1 Defective Dollar Bills

A dollar bill is *defective* if some digit appears more than once in the 8-digit serial number. If you check your wallet, you'll be sad to discover that defective bills are all-too-common. In fact, how common are *nondefective* bills? Assuming that the digit portions of serial numbers all occur equally often, we could answer this question by computing

$$\text{fraction of nondefective bills} = \frac{|\{\text{serial \#'s with all digits different}\}|}{|\{\text{serial numbers}\}|}. \tag{11.1}$$

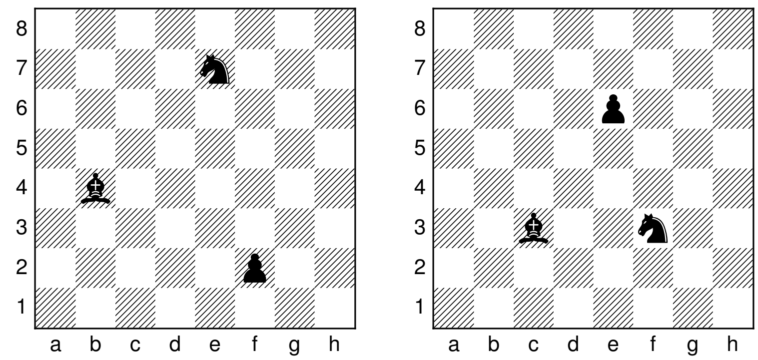
Let's first consider the denominator. Here there are no restrictions; there are are 10 possible first digits, 10 possible second digits, 10 third digits, and so on. Thus, the total number of 8-digit serial numbers is 10^8 by the Product Rule.

Next, let's turn to the numerator. Now we're not permitted to use any digit twice. So there are still 10 possible first digits, but only 9 possible second digits, 8 possible third digits, and so forth. Thus, by the Generalized Product Rule, there are

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = \frac{10!}{2} = 1{,}814{,}400$$

serial numbers with all digits different. Plugging these results into Equation 11.1, we find:

$$\text{fraction of nondefective bills} = \frac{1{,}814{,}400}{100{,}000{,}000} = 1.8144\%$$



(a) valid (b) invalid

Figure 11.1 Two ways of placing a pawn (♙), a knight (♘), and a bishop (♗) on a chessboard. The configuration shown in (b) is invalid because the bishop and the knight are in the same row.

11.3.2 A Chess Problem

In how many different ways can we place a pawn (P), a knight (N), and a bishop (B) on a chessboard so that no two pieces share a row or a column? A valid configuration is shown in Figure 11.1(a), and an invalid configuration is shown in Figure 11.1(b).

First, we map this problem about chess pieces to a question about sequences. There is a bijection from configurations to sequences

$$(r_P, c_P, r_N, c_N, r_B, c_B)$$

where r_P, r_N, and r_B are distinct rows and c_P, c_N, and c_B are distinct columns. In particular, r_P is the pawn's row, c_P is the pawn's column, r_N is the knight's row, etc. Now we can count the number of such sequences using the Generalized Product Rule:

- r_P is one of 8 rows
- c_P is one of 8 columns
- r_N is one of 7 rows (any one but r_P)
- c_N is one of 7 columns (any one but c_P)
- r_B is one of 6 rows (any one but r_P or r_N)
- c_B is one of 6 columns (any one but c_P or c_N)

Thus, the total number of configurations is $(8 \cdot 7 \cdot 6)^2$.

11.3.3 Permutations

A *permutation* of a set S is a sequence that contains every element of S exactly once. For example, here are all the permutations of the set $\{a, b, c\}$:

$$\begin{array}{ccc} (a, b, c) & (a, c, b) & (b, a, c) \\ (b, c, a) & (c, a, b) & (c, b, a) \end{array}$$

How many permutations of an n-element set are there? Well, there are n choices for the first element. For each of these, there are $n - 1$ remaining choices for the second element. For every combination of the first two elements, there are $n - 2$ ways to choose the third element, and so forth. Thus, there are a total of

$$n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1 = n!$$

permutations of an n-element set. In particular, this formula says that there are $3! = 6$ permutations of the 3-element set $\{a, b, c\}$, which is the number we found above.

Permutations will come up again in this course approximately 1.6 bazillion times. In fact, permutations are the reason why factorial comes up so often and why we taught you Stirling's approximation:

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

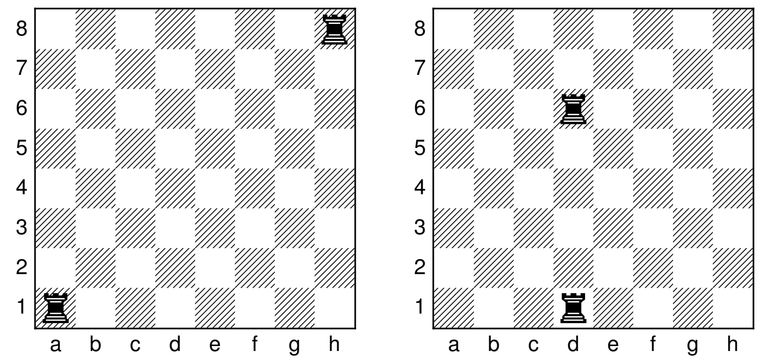
11.4 The Division Rule

Counting ears and dividing by two is a silly way to count the number of people in a room, but this approach is representative of a powerful counting principle.

A *k-to-1 function* maps exactly k elements of the domain to every element of the codomain. For example, the function mapping each ear to its owner is 2-to-1. Similarly, the function mapping each finger to its owner is 10-to-1, and the function mapping each finger and toe to its owner is 20-to-1. The general rule is:

Rule 11.4.1 (Division Rule)**.** *If $f : A \to B$ is k-to-1, then $|A| = k \cdot |B|$.*

For example, suppose A is the set of ears in the room and B is the set of people. There is a 2-to-1 mapping from ears to people, so by the Division Rule, $|A| = 2 \cdot |B|$. Equivalently, $|B| = |A|/2$, expressing what we knew all along: the number of people is half the number of ears. Unlikely as it may seem, many counting problems are made much easier by initially counting every item multiple times and then correcting the answer using the Division Rule. Let's look at some examples.



(a) valid (b) invalid

Figure 11.2 Two ways to place 2 rooks (♖) on a chessboard. The configuration in (b) is invalid because the rooks are in the same column.

11.4.1 Another Chess Problem

In how many different ways can you place two identical rooks on a chessboard so that they do not share a row or column? A valid configuration is shown in Figure 11.2(a), and an invalid configuration is shown in Figure 11.2(b).

Let A be the set of all sequences

$$(r_1, c_1, r_2, c_2)$$

where r_1 and r_2 are distinct rows and c_1 and c_2 are distinct columns. Let B be the set of all valid rook configurations. There is a natural function f from set A to set B; in particular, f maps the sequence (r_1, c_1, r_2, c_2) to a configuration with one rook in row r_1, column c_1 and the other rook in row r_2, column c_2.

But now there's a snag. Consider the sequences:

$$(1, 1, 8, 8) \qquad \text{and} \qquad (8, 8, 1, 1)$$

The first sequence maps to a configuration with a rook in the lower-left corner and a rook in the upper-right corner. The second sequence maps to a configuration with a rook in the upper-right corner and a rook in the lower-left corner. The problem is that those are two different ways of describing the *same* configuration! In fact, this arrangement is shown in Figure 11.2(a).

More generally, the function f maps exactly two sequences to *every* board configuration; that is f is a 2-to-1 function. Thus, by the quotient rule, $|A| = 2 \cdot |B|$.

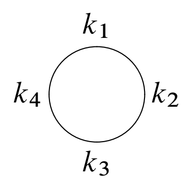
Rearranging terms gives:

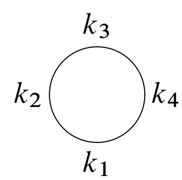
$$|B| = \frac{|A|}{2} = \frac{(8 \cdot 7)^2}{2}.$$

On the second line, we've computed the size of A using the General Product Rule just as in the earlier chess problem.

11.4.2 Knights of the Round Table

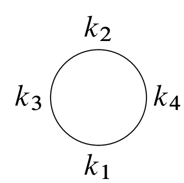
In how many ways can King Arthur seat n different knights at his round table? Two seatings are considered equivalent if one can be obtained from the other by rotation. For example, the following two arrangements are equivalent:





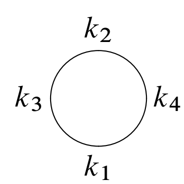
Let A be all the permutations of the knights, and let B be the set of all possible seating arrangements at the round table. We can map each permutation in set A to a circular seating arrangement in set B by seating the first knight in the permutation anywhere, putting the second knight to his left, the third knight to the left of the second, and so forth all the way around the table. For example:

$$(k_2, k_4, k_1, k_3) \longrightarrow$$



This mapping is actually an n-to-1 function from A to B, since all n cyclic shifts of the original sequence map to the same seating arrangement. In the example, $n = 4$ different sequences map to the same seating arrangement:

$$\begin{matrix}(k_2, k_4, k_1, k_3)\\(k_4, k_1, k_3, k_2)\\(k_1, k_3, k_2, k_4)\\(k_3, k_2, k_4, k_1)\end{matrix} \longrightarrow$$



Therefore, by the division rule, the number of circular seating arrangements is:

$$|B| = \frac{|A|}{n} = \frac{n!}{n} = (n-1)!$$

Note that $|A| = n!$ since there are $n!$ permutations of n knights.

11.5 Counting Subsets

How many k-element subsets of an n-element set are there? This question arises all the time in various guises:

- In how many ways can I select 5 books from my collection of 100 to bring on vacation?
- How many different 13-card Bridge hands can be dealt from a 52-card deck?
- In how many ways can I select 5 toppings for my pizza if there are 14 available toppings?

This number comes up so often that there is a special notation for it:

$$\binom{n}{k} ::= \text{the number of } k\text{-element subsets of an } n\text{-element set.}$$

The expression $\binom{n}{k}$ is read "n choose k." Now we can immediately express the answers to all three questions above:

- I can select 5 books from 100 in $\binom{100}{5}$ ways.
- There are $\binom{52}{13}$ different Bridge hands.
- There are $\binom{14}{5}$ different 5-topping pizzas, if 14 toppings are available.

11.5.1 The Subset Rule

We can derive a simple formula for the n-choose-k number using the Division Rule. We do this by mapping any permutation of an n-element set $\{a_1, \dots, a_n\}$ into a k-element subset simply by taking the first k elements of the permutation. That is, the permutation $a_1 a_2 \dots a_n$ will map to the set $\{a_1, a_2, \dots, a_k\}$.

Notice that any other permutation with the same first k elements $a_1, \dots, a_k$ *in any order* and the same remaining elements $n - k$ elements *in any order* will also map to this set. What's more, a permutation can only map to $\{a_1, a_2, \dots, a_k\}$ if its first k elements are the elements $a_1, \dots, a_k$ in some order. Since there are $k!$ possible permutations of the first k elements and $(n - k)!$ permutations of the remaining elements, we conclude from the Product Rule that exactly $k!(n - k)!$ permutations of the n-element set map to the the particular subset, S. In other words, the mapping from permutations to k-element subsets is $k!(n - k)!$-to-1.

But we know there are $n!$ permutations of an n-element set, so by the Division Rule, we conclude that

$$n! = k!(n-k)! \binom{n}{k}$$

which proves:

Rule 11.5.1 (Subset Rule)**.** *The number of k-element subsets of an n-element set is*

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

Notice that this works even for 0-element subsets: $n!/0!n! = 1$. Here we use the fact that 0! is a *product* of 0 terms, which by convention[2] equals 1.

11.5.2 Bit Sequences

How many n-bit sequences contain exactly k ones? We've already seen the straightforward bijection between subsets of an n-element set and n-bit sequences. For example, here is a 3-element subset of $\{x_1, x_2, \dots, x_8\}$ and the associated 8-bit sequence:

$$\begin{array}{ccccccccc} \{\, x_1, & & & x_4, & x_5 & & & & \} \\ (\; 1, & 0, & 0, & 1, & 1, & 0, & 0, & 0 &) \end{array}$$

Notice that this sequence has exactly 3 ones, each corresponding to an element of the 3-element subset. More generally, the n-bit sequences corresponding to a k-element subset will have exactly k ones. So by the Bijection Rule,

[2] We don't use it here, but a *sum* of zero terms equals 0.

The number of n-bit sequences with exactly k ones is $\binom{n}{k}$.

11.6 Sequences with Repetitions

11.6.1 Sequences of Subsets

Choosing a k-element subset of an n-element set is the same as splitting the set into a pair of subsets: the first subset of size k and the second subset consisting of the remaining $n - k$ elements. So the Subset Rule can be understood as a rule for counting the number of such splits into pairs of subsets.

We can generalize this to splits into more than two subsets. Namely, let A be an n-element set and $k_1, k_2, \ldots, k_m$ be nonnegative integers whose sum is n. A $(k_1, k_2, \ldots, k_m)$-*split of* A is a sequence

$$(A_1, A_2, \ldots, A_m)$$

where the A_i are disjoint subsets of A and $|A_i| = k_i$ for $i = 1, \ldots, m$.

Rule 11.6.1 (Subset Split Rule)**.** *The number of* $(k_1, k_2, \ldots, k_m)$*-splits of an* n*-element set is*

$$\binom{n}{k_1, \ldots, k_m} ::= \frac{n!}{k_1!\, k_2! \cdots k_m!}$$

The proof of this Rule is essentially the same as for the Subset Rule. Namely, we map any permutation $a_1 a_2 \ldots a_n$ of an n-element set A into a $(k_1, k_2, \ldots, k_m)$-split by letting the 1st subset in the split be the first k_1 elements of the permutation, the 2nd subset of the split be the next k_2 elements, ..., and the mth subset of the split be the final k_m elements of the permutation. This map is a $k_1!\, k_2! \cdots k_m!$-to-1 function from the $n!$ permutations to the $(k_1, k_2, \ldots, k_m)$-splits of A, and the Subset Split Rule now follows from the Division Rule.

11.6.2 The Bookkeeper Rule

We can also generalize our count of n-bit sequences with k ones to counting sequences of n letters over an alphabet with more than two letters. For example, how many sequences can be formed by permuting the letters in the 10-letter word BOOKKEEPER?

Notice that there are 1 B, 2 O's, 2 K's, 3 E's, 1 P, and 1 R in BOOKKEEPER. This leads to a straightforward bijection between permutations of BOOKKEEPER and

(1,2,2,3,1,1)-splits of $\{1, 2, \ldots, 10\}$. Namely, map a permutation to the sequence of sets of positions where each of the different letters occur.

For example, in the permutation BOOKKEEPER itself, the B is in the 1st position, the O's occur in the 2nd and 3rd positions, K's in 4th and 5th, the E's in the 6th, 7th and 9th, P in the 8th, and R is in the 10th position. So BOOKKEEPER maps to

$$(\{1\}, \{2, 3\}, \{4, 5\}, \{6, 7, 9\}, \{8\}, \{10\}).$$

From this bijection and the Subset Split Rule, we conclude that the number of ways to rearrange the letters in the word BOOKKEEPER is:

$$\frac{\overbrace{10!}^{\text{total letters}}}{\underbrace{1!}_{\text{B's}}\;\underbrace{2!}_{\text{O's}}\;\underbrace{2!}_{\text{K's}}\;\underbrace{3!}_{\text{E's}}\;\underbrace{1!}_{\text{P's}}\;\underbrace{1!}_{\text{R's}}}$$

This example generalizes directly to an exceptionally useful counting principle which we will call the

Rule 11.6.2 (Bookkeeper Rule)**.** *Let $l_1, \ldots, l_m$ be distinct elements. The number of sequences with k_1 occurrences of l_1, and k_2 occurrences of l_2, ..., and k_m occurrences of l_m is*

$$\frac{(k_1 + k_2 + \ldots + k_m)!}{k_1!\,k_2!\,\ldots\,k_m!}$$

For example, suppose you are planning a 20-mile walk, which should include 5 northward miles, 5 eastward miles, 5 southward miles, and 5 westward miles. How many different walks are possible?

There is a bijection between such walks and sequences with 5 N's, 5 E's, 5 S's, and 5 W's. By the Bookkeeper Rule, the number of such sequences is:

$$\frac{20!}{5!^4}.$$

11.6.3 The Binomial Theorem

Counting gives insight into one of the basic theorems of algebra. A *binomial* is a sum of two terms, such as $a + b$. Now consider its 4th power, $(a + b)^4$.

If we multiply out this 4th power expression completely, we get

$$\begin{aligned}(a+b)^4 \;=\leftarrow\; & aaaa \;+\!\leftarrow aaab \;+\!\leftarrow aaba \;+\!\leftarrow aabb \\ & +\!\leftarrow abaa \;+\!\leftarrow abab \;+\!\leftarrow abba \;+\!\leftarrow abbb \\ & +\!\leftarrow baaa \;+\!\leftarrow baab \;+\!\leftarrow baba \;+\!\leftarrow babb \\ & +\!\leftarrow bbaa \;+\!\leftarrow bbab \;+\!\leftarrow bbba \;+\!\leftarrow bbbb\end{aligned}$$

Notice that there is one term for every sequence of a's and b's. So there are 2^4 terms, and the number of terms with k copies of b and $n-k$ copies of a is:

$$\frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

by the Bookkeeper Rule. Hence, the coefficient of $a^{n-k}b^k$ is $\binom{n}{k}$. So for $n = 4$, this means:

$$(a+b)^4 = \binom{4}{0} \cdot a^4b^0 + \binom{4}{1} \cdot a^3b^1 + \binom{4}{2} \cdot a^2b^2 + \binom{4}{3} \cdot a^1b^3 + \binom{4}{4} \cdot a^0b^4$$

In general, this reasoning gives the Binomial Theorem:

Theorem 11.6.3 (Binomial Theorem)**.** *For all $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$:*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k$$

The expression $\binom{n}{k}$ is often called a "binomial coefficient" in honor of its appearance here.

This reasoning about binomials extends nicely to *multinomials*, which are sums of two or more terms. For example, suppose we wanted the coefficient of

$$bo^2k^2e^3pr$$

in the expansion of $(b+o+k+e+p+r)^{10}$. Each term in this expansion is a product of 10 variables where each variable is one of b, o, k, e, p, or r. Now, the coefficient of $bo^2k^2e^3pr$ is the number of those terms with exactly 1 b, 2 o's, 2 k's, 3 e's, 1 p, and 1 r. And the number of such terms is precisely the number of rearrangements of the word BOOKKEEPER:

$$\binom{10}{1,2,2,3,1,1} = \frac{10!}{1!\,2!\,2!\,3!\,1!\,1!}.$$

The expression on the left is called a "multinomial coefficient." This reasoning extends to a general theorem.

Definition 11.6.4. For $n, k_1, \ldots, k_m \in \mathbb{N}$, such that $k_1+k_2+\cdots+k_m = n$, define the *multinomial coefficient*

$$\binom{n}{k_1, k_2, \ldots, k_m} ::= \frac{n!}{k_1!\,k_2!\,\ldots k_m!}.$$

Theorem 11.6.5 (Multinomial Theorem)**.** *For all $n \in \mathbb{N}$,*

$$(z_1 + z_2 + \cdots + z_m)^n = \sum_{\substack{k_1,\ldots,k_m \in \mathbb{N} \\ k_1+\cdots+k_m=n}} \binom{n}{k_1, k_2, \ldots, k_m} z_1^{k_1} z_2^{k_2} \cdots z_m^{k_m}.$$

You'll be better off remembering the reasoning behind the Multinomial Theorem rather than this ugly formal statement.

11.6.4 A Word about Words

Someday you might refer to the Subset Split Rule or the Bookkeeper Rule in front of a roomful of colleagues and discover that they're all staring back at you blankly. This is not because they're dumb, but rather because we made up the name "Bookkeeper Rule". However, the rule is excellent and the name is apt, so we suggest that you play through: "You know? The Bookkeeper Rule? Don't you guys know *anything???*"

The Bookkeeper Rule is sometimes called the "formula for permutations with indistinguishable objects." The size k subsets of an n-element set are sometimes called *k-combinations*. Other similar-sounding descriptions are "combinations with repetition, permutations with repetition, r-permutations, permutations with indistinguishable objects," and so on. However, the counting rules we've taught you are sufficient to solve all these sorts of problems without knowing this jargon, so we won't burden you with it.

11.7 Counting Practice: Poker Hands

Five-Card Draw is a card game in which each player is initially dealt a *hand* consisting of 5 cards from a deck of 52 cards.[3] (Then the game gets complicated, but let's not worry about that.) The number of different hands in Five-Card Draw is the

[3]There are 52 cards in a standard deck. Each card has a *suit* and a *rank*. There are four suits:

$$\spadesuit \text{ (spades)} \quad \heartsuit \text{ (hearts)} \quad \clubsuit \text{ (clubs)} \quad \diamondsuit \text{ (diamonds)}$$

And there are 13 ranks, listed here from lowest to highest:

$$\overset{\text{Ace}}{A}, 2, 3, 4, 5, 6, 7, 8, 9, \overset{\text{Jack}}{J}, \overset{\text{Queen}}{Q}, \overset{\text{King}}{K}.$$

Thus, for example, $8\heartsuit$ is the 8 of hearts and $A\spadesuit$ is the ace of spades.

number of 5-element subsets of a 52-element set, which is

$$\binom{52}{5} = 2,598,960.$$

Let's get some counting practice by working out the number of hands with various special properties.

11.7.1 Hands with a Four-of-a-Kind

A *Four-of-a-Kind* is a set of four cards with the same rank. How many different hands contain a Four-of-a-Kind? Here are a couple examples:

$$\{8\spadesuit,\ 8\diamondsuit,\ Q\heartsuit,\ 8\heartsuit,\ 8\clubsuit\}$$
$$\{A\clubsuit,\ 2\clubsuit,\ 2\heartsuit,\ 2\diamondsuit,\ 2\spadesuit\}$$

As usual, the first step is to map this question to a sequence-counting problem. A hand with a Four-of-a-Kind is completely described by a sequence specifying:

1. The rank of the four cards.
2. The rank of the extra card.
3. The suit of the extra card.

Thus, there is a bijection between hands with a Four-of-a-Kind and sequences consisting of two distinct ranks followed by a suit. For example, the three hands above are associated with the following sequences:

$$(8, Q, \heartsuit) \leftrightarrow \{\,8\spadesuit,\ 8\diamondsuit,\ 8\heartsuit,\ 8\clubsuit,\ Q\heartsuit\}$$
$$(2, A, \clubsuit) \leftrightarrow \{2\clubsuit,\ 2\heartsuit,\ 2\diamondsuit,\ 2\spadesuit,\ A\clubsuit\}$$

Now we need only count the sequences. There are 13 ways to choose the first rank, 12 ways to choose the second rank, and 4 ways to choose the suit. Thus, by the Generalized Product Rule, there are $13 \cdot 12 \cdot 4 = 624$ hands with a Four-of-a-Kind. This means that only 1 hand in about 4165 has a Four-of-a-Kind. Not surprisingly, Four-of-a-Kind is considered to be a very good poker hand!

11.7.2 Hands with a Full House

A *Full House* is a hand with three cards of one rank and two cards of another rank. Here are some examples:

$$\{2\spadesuit, 2\clubsuit, 2\diamondsuit, J\clubsuit, J\diamondsuit\}$$
$$\{5\diamondsuit, 5\clubsuit, 5\heartsuit, 7\heartsuit, 7\clubsuit\}$$

Again, we shift to a problem about sequences. There is a bijection between Full Houses and sequences specifying:

1. The rank of the triple, which can be chosen in 13 ways.
2. The suits of the triple, which can be selected in $\binom{4}{3}$ ways.
3. The rank of the pair, which can be chosen in 12 ways.
4. The suits of the pair, which can be selected in $\binom{4}{2}$ ways.

The example hands correspond to sequences as shown below:

$$(2, \{\spadesuit, \clubsuit, \diamondsuit\}, J, \{\clubsuit, \diamondsuit\}) \leftrightarrow \{2\spadesuit, 2\clubsuit, 2\diamondsuit, J\clubsuit, J\diamondsuit\}$$
$$(5, \{\diamondsuit, \clubsuit, \heartsuit\}, 7, \{\heartsuit, \clubsuit\}) \leftrightarrow \{5\diamondsuit, 5\clubsuit, 5\heartsuit, 7\heartsuit, 7\clubsuit\}$$

By the Generalized Product Rule, the number of Full Houses is:

$$13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}.$$

We're on a roll—but we're about to hit a speed bump.

11.7.3 Hands with Two Pairs

How many hands have *Two Pairs*; that is, two cards of one rank, two cards of another rank, and one card of a third rank? Here are examples:

$$\{3\diamondsuit, 3\spadesuit, Q\diamondsuit, Q\heartsuit, A\clubsuit\}$$
$$\{9\heartsuit, 9\diamondsuit, 5\heartsuit, 5\clubsuit, K\spadesuit\}$$

Each hand with Two Pairs is described by a sequence consisting of:

1. The rank of the first pair, which can be chosen in 13 ways.
2. The suits of the first pair, which can be selected $\binom{4}{2}$ ways.

3. The rank of the second pair, which can be chosen in 12 ways.
4. The suits of the second pair, which can be selected in $\binom{4}{2}$ ways.
5. The rank of the extra card, which can be chosen in 11 ways.
6. The suit of the extra card, which can be selected in $\binom{4}{1} = 4$ ways.

Thus, it might appear that the number of hands with Two Pairs is:

$$13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 11 \cdot 4.$$

Wrong answer! The problem is that there is *not* a bijection from such sequences to hands with Two Pairs. This is actually a 2-to-1 mapping. For example, here are the pairs of sequences that map to the hands given above:

$$\begin{array}{rl} (3, \{\diamondsuit, \spadesuit\}, Q, \{\diamondsuit, \heartsuit\}, A, \clubsuit) & \searrow\leftarrow \\ & \{3\diamondsuit,\ 3\spadesuit,\ Q\diamondsuit,\ Q\heartsuit,\ A\clubsuit\} \\ (Q, \{\diamondsuit, \heartsuit\}, 3, \{\diamondsuit, \spadesuit\}, A, \clubsuit) & \nearrow\leftarrow \\ \\ (9, \{\heartsuit, \diamondsuit\}, 5, \{\heartsuit, \clubsuit\}, K, \spadesuit) & \searrow\leftarrow \\ & \{9\heartsuit,\ 9\diamondsuit,\ 5\heartsuit,\ 5\clubsuit,\ K\spadesuit\} \\ (5, \{\heartsuit, \clubsuit\}, 9, \{\heartsuit, \diamondsuit\}, K, \spadesuit) & \nearrow\leftarrow \end{array}$$

The problem is that nothing distinguishes the first pair from the second. A pair of 5’s and a pair of 9’s is the same as a pair of 9’s and a pair of 5’s. We avoided this difficulty in counting Full Houses because, for example, a pair of 6’s and a triple of kings is different from a pair of kings and a triple of 6’s.

We ran into precisely this difficulty last time, when we went from counting arrangements of *different* pieces on a chessboard to counting arrangements of two *identical* rooks. The solution then was to apply the Division Rule, and we can do the same here. In this case, the Division rule says there are twice as many sequences as hands, so the number of hands with Two Pairs is actually:

$$\frac{13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 11 \cdot 4}{2}.$$

Another Approach

The preceding example was disturbing! One could easily overlook the fact that the mapping was 2-to-1 on an exam, fail the course, and turn to a life of crime. You can make the world a safer place in two ways:

1. Whenever you use a mapping $f : A \to B$ to translate one counting problem to another, check that the same number elements in A are mapped to each element in B. If k elements of A map to each of element of B, then apply the Division Rule using the constant k.

2. As an extra check, try solving the same problem in a different way. Multiple approaches are often available—and all had better give the same answer! (Sometimes different approaches give answers that *look* different, but turn out to be the same after some algebra.)

We already used the first method; let's try the second. There is a bijection between hands with two pairs and sequences that specify:

1. The ranks of the two pairs, which can be chosen in $\binom{13}{2}$ ways.
2. The suits of the lower-rank pair, which can be selected in $\binom{4}{2}$ ways.
3. The suits of the higher-rank pair, which can be selected in $\binom{4}{2}$ ways.
4. The rank of the extra card, which can be chosen in 11 ways.
5. The suit of the extra card, which can be selected in $\binom{4}{1} = 4$ ways.

For example, the following sequences and hands correspond:

$$(\{3, Q\}, \{\diamondsuit, \spadesuit\}, \{\diamondsuit, \heartsuit\}, A, \clubsuit) \leftrightarrow \{3\diamondsuit,\ 3\spadesuit,\ Q\diamondsuit,\ Q\heartsuit,\ A\clubsuit\}$$
$$(\{9, 5\}, \{\heartsuit, \clubsuit\}, \{\heartsuit, \diamondsuit\}, K, \spadesuit) \leftrightarrow \{9\heartsuit,\ 9\diamondsuit,\ 5\heartsuit,\ 5\clubsuit,\ K\spadesuit\}$$

Thus, the number of hands with two pairs is:

$$\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 11 \cdot 4.$$

This is the same answer we got before, though in a slightly different form.

11.7.4 Hands with Every Suit

How many hands contain at least one card from every suit? Here is an example of such a hand:

$$\{7\diamondsuit,\ K\clubsuit,\ 3\diamondsuit,\ A\heartsuit,\ 2\spadesuit\}$$

Each such hand is described by a sequence that specifies:

1. The ranks of the diamond, the club, the heart, and the spade, which can be selected in $13 \cdot 13 \cdot 13 \cdot 13 = 13^4$ ways.

2. The suit of the extra card, which can be selected in 4 ways.

3. The rank of the extra card, which can be selected in 12 ways.

For example, the hand above is described by the sequence:

$$(7, K, A, 2, \diamondsuit, 3) \leftrightarrow \{7\diamondsuit,\ K\clubsuit,\ A\heartsuit,\ 2\spadesuit,\ 3\diamondsuit\}.$$

Are there other sequences that correspond to the same hand? There is one more! We could equally well regard either the $3\diamondsuit$ or the $7\diamondsuit$ as the extra card, so this is actually a 2-to-1 mapping. Here are the two sequences corresponding to the example hand:

$$\begin{array}{ll} (7, K, A, 2, \diamondsuit, 3) & \searrow\leftarrow \\ & \qquad \{7\diamondsuit,\ K\clubsuit,\ A\heartsuit,\ 2\spadesuit,\ 3\diamondsuit\} \\ (3, K, A, 2, \diamondsuit, 7) & \nearrow\leftarrow \end{array}$$

Therefore, the number of hands with every suit is:

$$\frac{13^4 \cdot 4 \cdot 12}{2}.$$

11.8 Inclusion-Exclusion

How big is a union of sets? For example, suppose there are 60 math majors, 200 EECS majors, and 40 physics majors. How many students are there in these three departments? Let M be the set of math majors, E be the set of EECS majors, and P be the set of physics majors. In these terms, we're asking for $|M \cup E \cup P|$.

The Sum Rule says that if M, E, and P are disjoint, then the sum of their sizes is

$$|M \cup E \cup P| = |M| + |E| + |P|.$$

However, the sets M, E, and P might *not* be disjoint. For example, there might be a student majoring in both math and physics. Such a student would be counted twice on the right side of this equation, once as an element of M and once as an element of P. Worse, there might be a triple-major[4] counted *three* times on the right side!

Our most-complicated counting rule determines the size of a union of sets that are not necessarily disjoint. Before we state the rule, let's build some intuition by considering some easier special cases: unions of just two or three sets.

[4]. . . though not at MIT anymore.

11.8.1 Union of Two Sets

For two sets, S_1 and S_2, the *Inclusion-Exclusion Rule* is that the size of their union is:

$$|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2| \qquad (11.2)$$

Intuitively, each element of S_1 is accounted for in the first term, and each element of S_2 is accounted for in the second term. Elements in *both* S_1 and S_2 are counted *twice*—once in the first term and once in the second. This double-counting is corrected by the final term.

11.8.2 Union of Three Sets

So how many students are there in the math, EECS, and physics departments? In other words, what is $|M \cup E \cup P|$ if:

$$\begin{aligned} |M| &= 60 \\ |E| &= 200 \\ |P| &= 40. \end{aligned}$$

The size of a union of three sets is given by a more complicated Inclusion-Exclusion formula:

$$\begin{aligned} |S_1 \cup S_2 \cup S_3| = {} & |S_1| + |S_2| + |S_3| \\ & - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| \\ & + |S_1 \cap S_2 \cap S_3|. \end{aligned}$$

Remarkably, the expression on the right accounts for each element in the union of S_1, S_2, and S_3 exactly once. For example, suppose that x is an element of all three sets. Then x is counted three times (by the $|S_1|$, $|S_2|$, and $|S_3|$ terms), subtracted off three times (by the $|S_1 \cap S_2|$, $|S_1 \cap S_3|$, and $|S_2 \cap S_3|$ terms), and then counted once more (by the $|S_1 \cap S_2 \cap S_3|$ term). The net effect is that x is counted just once.

If x is in two sets (say, S_1 and S_2), then x is counted twice (by the $|S_1|$ and $|S_2|$ terms) and subtracted once (by the $|S_1 \cap S_2|$ term). In this case, x does not factor into any of the other terms, since $x \notin S_3$.

So we can't answer the original question without knowing the sizes of the various intersections. Let's suppose that there are:

4	math - EECS double majors
3	math - physics double majors
11	EECS - physics double majors
2	triple majors

Then $|M \cap E| = 4+2$, $|M \cap P| = 3+2$, $|E \cap P| = 11+2$, and $|M \cap E \cap P| = 2$. Plugging all this into the formula gives:

$$\begin{aligned} |M \cup E \cup P| &= |M| + |E| + |P| - |M \cap E| - |M \cap P| - |E \cap P| + |M \cap E \cap P| \\ &= 60 + 200 + 40 - 6 - 5 - 13 + 2 \\ &= 278 \end{aligned}$$

11.8.3 Sequences with 42, 04, or 60

In how many permutations of the set $\{0, 1, 2, \ldots, 9\}$ do either 4 and 2, 0 and 4, or 6 and 0 appear consecutively? For example, none of these pairs appears in:

$$(7, 2, 9, 5, 4, 1, 3, 8, 0, 6).$$

The 06 at the end doesn't count; we need 60. On the other hand, both 04 and 60 appear consecutively in this permutation:

$$(7, 2, 5, \underline{6}, \underline{0}, \underline{4}, 3, 8, 1, 9).$$

Let P_{42} be the set of all permutations in which 42 appears. Define P_{60} and P_{04} similarly. Thus, for example, the permutation above is contained in both P_{60} and P_{04}, but not P_{42}. In these terms, we're looking for the size of the set $P_{42} \cup P_{04} \cup P_{60}$.

First, we must determine the sizes of the individual sets, such as P_{60}. We can use a trick: group the 6 and 0 together as a single symbol. Then there is a natural bijection between permutations of $\{0, 1, 2, \ldots 9\}$ containing 6 and 0 consecutively and permutations of:

$$\{60, 1, 2, 3, 4, 5, 7, 8, 9\}.$$

For example, the following two sequences correspond:

$$(7, 2, 5, \underline{6}, \underline{0}, 4, 3, 8, 1, 9) \quad \longleftrightarrow \quad (7, 2, 5, \underline{60}, 4, 3, 8, 1, 9).$$

There are 9! permutations of the set containing 60, so $|P_{60}| = 9!$ by the Bijection Rule. Similarly, $|P_{04}| = |P_{42}| = 9!$ as well.

Next, we must determine the sizes of the two-way intersections, such as $P_{42} \cap P_{60}$. Using the grouping trick again, there is a bijection with permutations of the set:

$$\{42, 60, 1, 3, 5, 7, 8, 9\}.$$

Thus, $|P_{42} \cap P_{60}| = 8!$. Similarly, $|P_{60} \cap P_{04}| = 8!$ by a bijection with the set:

$$\{604, 1, 2, 3, 5, 7, 8, 9\}.$$

And $|P_{42} \cap P_{04}| = 8!$ as well by a similar argument. Finally, note that $|P_{60} \cap P_{04} \cap P_{42}| = 7!$ by a bijection with the set:

$$\{6042, 1, 3, 5, 7, 8, 9\}.$$

Plugging all this into the formula gives:

$$|P_{42} \cup P_{04} \cup P_{60}| = 9! + 9! + 9! - 8! - 8! - 8! + 7!.$$

11.8.4 Union of *n* Sets

The size of a union of n sets is given by the following rule.

Rule 11.8.1 (Inclusion-Exclusion)**.**

$$|S_1 \cup S_2 \cup \cdots \cup S_n| =$$

	the sum of the sizes of the individual sets
minus	*the sizes of all two-way intersections*
plus	*the sizes of all three-way intersections*
minus	*the sizes of all four-way intersections*
plus	*the sizes of all five-way intersections, etc.*

The formulas for unions of two and three sets are special cases of this general rule.

This way of expressing Inclusion-Exclusion is easy to understand and nearly as precise as expressing it in mathematical symbols, but we'll need the symbolic version below, so let's work on deciphering it now.

We already have a standard notation for the sum of sizes of the individual sets, namely,

$$\sum_{i=1}^{n} |S_i|.$$

A "two-way intersection" is a set of the form $S_i \cap S_j$ for $i \neq j$. We regard $S_j \cap S_i$ as the same two-way intersection as $S_i \cap S_j$, so we can assume that $i < j$. Now we can express the sum of the sizes of the two-way intersections as

$$\sum_{1 \le i < j \le n} |S_i \cap S_j|.$$

Similarly, the sum of the sizes of the three-way intersections is

$$\sum_{1 \le i < j < k \le n} |S_i \cap S_j \cap S_k|.$$

These sums have alternating signs in the Inclusion-Exclusion formula, with the sum of the k-way intersections getting the sign $(-1)^{k-1}$. This finally leads to a symbolic version of the rule:

Rule (Inclusion-Exclusion)**.**

$$\begin{aligned}\left|\bigcup_{i=1}^{n} S_i\right| = &\sum_{i=1}^{n} |S_i| \\ &- \sum_{1\le i<j\le n} |S_i \cap S_j| \\ &+ \sum_{1\le i<j<k\le n} |S_i \cap S_j \cap S_k| + \cdots \\ &+ (-1)^{n-1}\left|\bigcap_{i=1}^{n} S_i\right|.\end{aligned}$$

11.8.5 Computing Euler's Function

As an example, let's use Inclusion-Exclusion to calculate Euler's function, $\phi(n)$. By definition, $\phi(n)$ is the number of nonnegative integers less than a positive integer n that are relatively prime to n. But the set S of nonnegative integers less than n that are *not* relatively prime to n will be easier to count.

Suppose the prime factorization of n is $p_1^{e_1} \cdots p_m^{e_m}$ for distinct primes p_i. This means that the integers in S are precisely the nonnegative integers less than n that are divisible by at least one of the p_i's. Letting C_i be the set of nonnegative integers less than n that are divisible by p_i, we have

$$S = \bigcup_{i=1}^{m} C_i.$$

We'll be able to find the size of this union using Inclusion-Exclusion because the intersections of the C_i's are easy to count. For example, $C_1 \cap C_2 \cap C_3$ is the set of nonnegative integers less than n that are divisible by each of p_1, p_2 and p_3. But since the p_i's are distinct primes, being divisible by each of these primes is the same as being divisible by their product. Now observe that if r is a positive divisor of n, then exactly n/r nonnegative integers less than n are divisible by r, namely, $0, r, 2r, \ldots, ((n/r)-1)r$. So exactly $n/p_1p_2p_3$ nonnegative integers less than n are divisible by all three primes p_1, p_2, p_3. In other words,

$$|C_1 \cap C_2 \cap C_3| = \frac{n}{p_1p_2p_3}.$$

Reasoning this way about all the intersections among the C_i's and applying Inclusion-Exclusion, we get

$$\begin{aligned}
|S| &= \left|\bigcup_{i=1}^{m} C_i\right| \\
&= \sum_{i=1}^{m} |C_i| - \sum_{1 \le i < j \le m} |C_i \cap C_j| + \sum_{1 \le i < j < k \le m} |C_i \cap C_j \cap C_k| - \cdots + (-1)^{m-1} \left|\bigcap_{i=1}^{m} C_i\right| \\
&= \sum_{i=1}^{m} \frac{n}{p_i} - \sum_{1 \le i < j \le m} \frac{n}{p_i p_j} + \sum_{1 \le i < j < k \le m} \frac{n}{p_i p_j p_k} - \cdots + (-1)^{m-1} \frac{n}{p_1 p_2 \cdots p_n} \\
&= n \left(\sum_{i=1}^{m} \frac{1}{p_i} - \sum_{1 \le i < j \le m} \frac{1}{p_i p_j} + \sum_{1 \le i < j < k \le m} \frac{1}{p_i p_j p_k} - \cdots + (-1)^{m-1} \frac{1}{p_1 p_2 \cdots p_n} \right)
\end{aligned}$$

But $\phi(n) = n - |S|$ by definition, so

$$\begin{aligned}
\phi(n) &= n \left(1 - \sum_{i=1}^{m} \frac{1}{p_i} + \sum_{1 \le i < j \le m} \frac{1}{p_i p_j} - \sum_{1 \le i < j < k \le m} \frac{1}{p_i p_j p_k} + \cdots + (-1)^{m} \frac{1}{p_1 p_2 \cdots p_n} \right) \\
&= n \prod_{i=1}^{m} \left(1 - \frac{1}{p_i} \right). \qquad (11.3)
\end{aligned}$$

Yikes! That was pretty hairy. Are you getting tired of all that nasty algebra? If so, then good news is on the way. In the next section, we will show you how to prove some heavy-duty formulas without using any algebra at all. Just a few words and you are done. No kidding.

11.9 Combinatorial Proofs

Suppose you have n different T-shirts, but only want to keep k. You could equally well select the k shirts you want to keep or select the complementary set of $n - k$ shirts you want to throw out. Thus, the number of ways to select k shirts from among n must be equal to the number of ways to select $n - k$ shirts from among n. Therefore:

$$\binom{n}{k} = \binom{n}{n-k}.$$

This is easy to prove algebraically, since both sides are equal to:

$$\frac{n!}{k!\,(n-k)!}.$$

But we didn't really have to resort to algebra; we just used counting principles.

Hmmm. . .

11.9.1 Pascal's Identity

Jay, famed Math for Computer Science Teaching Assistant, has decided to try out for the US Olympic boxing team. After all, he's watched all of the *Rocky* movies and spent hours in front of a mirror sneering, "Yo, you wanna piece a' *me*?!" Jay figures that n people (including himself) are competing for spots on the team and only k will be selected. As part of maneuvering for a spot on the team, he needs to work out how many different teams are possible. There are two cases to consider:

- Jay *is* selected for the team, and his $k-1$ teammates are selected from among the other $n-1$ competitors. The number of different teams that can be formed in this way is:

$$\binom{n-1}{k-1}.$$

- Jay is *not* selected for the team, and all k team members are selected from among the other $n-1$ competitors. The number of teams that can be formed this way is:

$$\binom{n-1}{k}.$$

All teams of the first type contain Jay, and no team of the second type does; therefore, the two sets of teams are disjoint. Thus, by the Sum Rule, the total number of possible Olympic boxing teams is:

$$\binom{n-1}{k-1}+\binom{n-1}{k}.$$

Jeremy, equally-famed Teaching Assistant, thinks Jay isn't so tough and so he might as well also try out. He reasons that n people (including himself) are trying out for k spots. Thus, the number of ways to select the team is simply:

$$\binom{n}{k}.$$

Jeremy and Jay each correctly counted the number of possible boxing teams. Thus, their answers must be equal. So we know:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

This is called *Pascal's Identity*. And we proved it *without any algebra!* Instead, we relied purely on counting techniques.

11.9.2 Finding a Combinatorial Proof

A *combinatorial proof* is an argument that establishes an algebraic fact by relying on counting principles. Many such proofs follow the same basic outline:

1. Define a set S.
2. Show that $|S| = n$ by counting one way.
3. Show that $|S| = m$ by counting another way.
4. Conclude that $n = m$.

In the preceding example, S was the set of all possible Olympic boxing teams. Jay computed

$$|S| = \binom{n-1}{k-1} + \binom{n-1}{k}$$

by counting one way, and Jeremy computed

$$|S| = \binom{n}{k}$$

by counting another way. Equating these two expressions gave Pascal's Identity.

More typically, the set S is defined in terms of simple sequences or sets rather than an elaborate story. Here is a less colorful example of a combinatorial argument.

Theorem 11.9.1.

$$\sum_{r=0}^{n} \binom{n}{r}\binom{2n}{n-r} = \binom{3n}{n}$$

Proof. We give a combinatorial proof. Let S be all n-card hands that can be dealt from a deck containing n red cards (numbered $1, \ldots, n$) and $2n$ black cards (numbered $1, \ldots, 2n$). First, note that every $3n$-element set has

$$|S| = \binom{3n}{n}$$

n-element subsets.

From another perspective, the number of hands with exactly r red cards is

$$\binom{n}{r}\binom{2n}{n-r}$$

since there are $\binom{n}{r}$ ways to choose the r red cards and $\binom{2n}{n-r}$ ways to choose the $n-r$ black cards. Since the number of red cards can be anywhere from 0 to n, the total number of n-card hands is:

$$|S| = \sum_{r=0}^{n} \binom{n}{r}\binom{2n}{n-r}.$$

Equating these two expressions for $|S|$ proves the theorem. ■

Combinatorial proofs are almost magical. Theorem 11.9.1 looks pretty scary, but we proved it without any algebraic manipulations at all. The key to constructing a combinatorial proof is choosing the set S properly, which can be tricky. Generally, the simpler side of the equation should provide some guidance. For example, the right side of Theorem 11.9.1 is $\binom{3n}{n}$, which suggests that it will be helpful to choose S to be all n-element subsets of some $3n$-element set.

11.10 The Pigeonhole Principle

Here is an old puzzle:

> A drawer in a dark room contains red socks, green socks, and blue socks. How many socks must you withdraw to be sure that you have a matching pair?

For example, picking out three socks is not enough; you might end up with one red, one green, and one blue. The solution relies on the *Pigeonhole Principle*, which is a friendly name for the contrapositive of the injective case of the Mapping Rule.

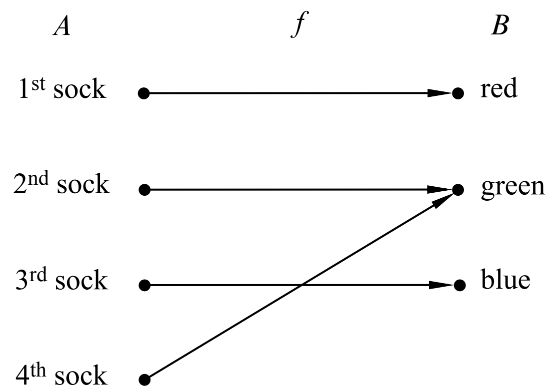


Figure 11.3 One possible mapping of four socks to three colors.

Rule 11.10.1 (Pigeonhole Principle)**.** *If $|X| > |Y|$, then for every total function*[5] *$f : X \to Y$, there exist two different elements of X that are mapped to the same element of Y.*

What this abstract mathematical statement has to do with selecting footwear under poor lighting conditions is maybe not obvious. However, let A be the set of socks you pick out, let B be the set of colors available, and let f map each sock to its color. The Pigeonhole Principle says that if $|A| > |B| = 3$, then at least two elements of A (that is, at least two socks) must be mapped to the same element of B (that is, the same color). Therefore, four socks are enough to ensure a matched pair. For example, one possible mapping of four socks to three colors is shown in Figure 11.3.

Not surprisingly, the pigeonhole principle is often described in terms of pigeons:

> *If there are more pigeons than holes they occupy, then at least two pigeons must be in the same hole.*

In this case, the pigeons form set A, the pigeonholes are set B, and f describes which hole each pigeon flies into.

Mathematicians have come up with many ingenious applications for the pigeonhole principle. If there were a cookbook procedure for generating such arguments, we'd give it to you. Unfortunately, there isn't one. One helpful tip, though: when you try to solve a problem with the pigeonhole principle, the key is to clearly identify three things:

[5]This Mapping Rule applies even if f is a total injective *relation*. Recall that a function is total if $\forall x \in X\, \exists y \in Y.\ f(x) = y$.

1. The set A (the pigeons).
2. The set B (the pigeonholes).
3. The function f (the rule for assigning pigeons to pigeonholes).

11.10.1 Hairs on Heads

There are a number of generalizations of the pigeonhole principle. For example:

Rule 11.10.2 (Generalized Pigeonhole Principle)**.** *If $|X| > k \cdot |Y|$, then every total function $f : X \to Y$ maps at least $k + 1$ different elements of X to the same element of Y.*

For example, if you pick two people at random, surely they are extremely unlikely to have *exactly* the same number of hairs on their heads. However, in the remarkable city of Boston, Massachusetts there are actually *three* people who have exactly the same number of hairs! Of course, there are many bald people in Boston, and they all have zero hairs. But we're talking about non-bald people; say a person is non-bald if they have at least ten thousand hairs on their head.

Boston has about 500,000 non-bald people, and the number of hairs on a person's head is at most 200,000. Let A be the set of non-bald people in Boston, let $B = \{10,000, 10,001, \ldots, 200,000\}$, and let f map a person to the number of hairs on his or her head. Since $|A| > 2|B|$, the Generalized Pigeonhole Principle implies that at least three people have exactly the same number of hairs. We don't know who they are, but we know they exist!

11.10.2 Subsets with the Same Sum

For your reading pleasure, we have displayed ninety 25-digit numbers in Figure 11.4. Are there two different subsets of these 25-digit numbers that have the same sum? For example, maybe the sum of the last ten numbers in the first column is equal to the sum of the first eleven numbers in the second column?

Finding two subsets with the same sum may seem like a silly puzzle, but solving these sorts of problems turns out to be useful in diverse applications such as finding good ways to fit packages into shipping containers and decoding secret messages.

It turns out that it is hard to find different subsets with the same sum, which is why this problem arises in cryptography. But it is easy to prove that two such subsets *exist*. That's where the Pigeonhole Principle comes in.

Let A be the collection of all subsets of the 90 numbers in the list. Now the sum of any subset of numbers is at most $90 \cdot 10^{25}$, since there are only 90 numbers and every 25-digit number is less than 10^{25}. So let B be the set of integers $\{0, 1, \ldots, 90 \cdot 10^{25}\}$, and let f map each subset of numbers (in A) to its sum (in B).

0020480135385502964448038	3171004832173501394113017
5763257331083479647409398	8247331000042995311646021
0489445991866915676240992	3208234421597368647019265
5800949123548989122628663	8496243997123475922766310
1082662032430379651370981	3437254656355157864869113
6042900801199280218026001	8518399140676002660747477
1178480894769706178994993	3574883393058653923711365
6116171789137737896701405	8543691283470191452333763
1253127351683239693851327	3644909946040480189969149
6144868973001582369723512	8675309258374137092461352
1301505129234077811069011	3790044132737084094417246
6247314593851169234746152	8694321112363996867296665
1311567111143866433882194	3870332127437971355322815
6814428944266874963488274	8772321203608477245851154
1470029452721203587686214	4080505804577801451363100
6870852945543886849147881	8791422161722582546341091
1578271047286257499433886	4167283461025702348124920
6914955508120950093732397	9062628024592126283973285
1638243921852176243192354	4235996831123777788211249
6949632451365987152423541	9137845566925526349897794
1763580219131985963102365	4670939445749439042111220
7128211143613619828415650	9153762966803189291934419
1826227795601842231029694	4815379351865384279613427
7173920083651862307925394	9270880194077636406984249
1843971862675102037201420	4837052948212922604442190
7215654874211755676220587	9324301480722103490379204
2396951193722134526177237	5106389423855018550671530
7256932847164391040233050	9436090832146695147140581
2781394568268599801096354	5142368192004769218069910
7332822657075235431620317	9475308159734538249013238
2796605196713610405408019	5181234096130144084041856
7426441829541573444964139	9492376623917486974923202
2931016394761975263190347	5198267398125617994391348
7632198126531809327186321	9511972558779880288252979
2933458058294405155197296	5317592940316231219758372
7712154432211912882310511	9602413424619187112552264
3075514410490975920315348	5384358126771794128356947
7858918664240262356610010	9631217114906129219461111
8149436716871371161932035	3157693105325111284321993
3111474985252793452860017	5439211712248901995423441
7898156786763212963178679	9908189853102753335981319
3145621587936120118438701	5610379826092838192760458
8147591017037573337848616	9913237476341764299813987
3148901255628881103198549	5632317555465228677676044
5692168374637019617423712	8176063831682536571306791

Figure 11.4 Ninety 25-digit numbers. Can you find two different subsets of these numbers that have the same sum?

We proved that an n-element set has 2^n different subsets in Section 11.2. Therefore:

$$|A| = 2^{90} \geq 1.237 \times 10^{27}$$

On the other hand:

$$|B| = 90 \cdot 10^{25} + 1 \leq 0.901 \times 10^{27}.$$

Both quantities are enormous, but $|A|$ is a bit greater than $|B|$. This means that f maps at least two elements of A to the same element of B. In other words, by the Pigeonhole Principle, two different subsets must have the same sum!

Notice that this proof gives no indication *which* two sets of numbers have the same sum. This frustrating variety of argument is called a *nonconstructive proof*. To see if was possible to actually *find* two different subsets of the ninety 25-digit numbers with the same sum, we offered a $100 prize to the first student who did it. We didn't expect to have to pay off this bet, but we underestimated the ingenuity and initiative of the students. One computer science major wrote a program that cleverly searched only among a reasonably small set of "plausible" sets, sorted them by their sums, and actually found a couple with the same sum. He won the prize. A few days later, a math major figured out how to reformulate the sum problem as a "lattice basis reduction" problem; then he found a software package implementing an efficient basis reduction procedure, and using it, he very quickly found lots of pairs of subsets with the same sum. He didn't win the prize, but he got a standing ovation from the class—staff included.

11.11 A Magic Trick

There is a Magician and an Assistant. The Assistant goes into the audience with a deck of 52 cards while the Magician looks away.

Five audience members each select one card from the deck. The Assistant then gathers up the five cards and holds up four of them so the Magician can see them. The Magician concentrates for a short time and then correctly names the secret, fifth card!

Since we don't really believe the Magician can read minds, we know the Assistant has somehow communicated the secret card to the Magician. Since real Magicians and Assistants are not to be trusted, we can expect that the Assistant would illegitimately signal the Magician with coded phrases or body language, but they don't have to cheat in this way. In fact, the Magician and Assistant could be

Sets with Distinct Subset Sums

How can we construct a set of n positive integers such that all its subsets have *distinct* sums? One way is to use powers of two:

$$\{1, 2, 4, 8, 16\}$$

This approach is so natural that one suspects all other such sets must involve larger numbers. (For example, we could safely replace 16 by 17, but not by 15.) Remarkably, there are examples involving *smaller* numbers. Here is one:

$$\{6, 9, 11, 12, 13\}$$

One of the top mathematicians of the Twentieth Century, Paul Erdős, conjectured in 1931 that there are no such sets involving *significantly* smaller numbers. More precisely, he conjectured that the largest number in such a set must be greater than $c2^n$ for some constant $c > 0$. He offered $500 to anyone who could prove or disprove his conjecture, but the problem remains unsolved.

kept out of sight of each other while some audience member holds up the 4 cards designated by the Assistant for the Magician to see.

Of course, without cheating, there is still an obvious way the Assistant can communicate to the Magician: he can choose any of the $4! = 24$ permutations of the 4 cards as the order in which to hold up the cards. However, this alone won't quite work: there are 48 cards remaining in the deck, so the Assistant doesn't have enough choices of orders to indicate exactly what the secret card is (though he could narrow it down to two cards).

11.11.1 The Secret

The method the Assistant can use to communicate the fifth card exactly is a nice application of what we know about counting and matching.

The Assistant has a second legitimate way to communicate: he can choose *which of the five cards to keep hidden*. Of course, it's not clear how the Magician could determine which of these five possibilities the Assistant selected by looking at the four visible cards, but there is a way, as we'll now explain.

The problem facing the Magician and Assistant is actually a bipartite matching problem. Put all the *sets* of 5 cards in a collection X on the left. And put all the *sequences* of 4 distinct cards in a collection Y on the right. These are the two sets of vertices in the bipartite graph. There is an edge between a set of 5 cards and a sequence of 4 if every card in the sequence is also in the set. In other words, if the audience selects a set of 5 cards, then the Assistant must reveal a sequence of 4 cards that is adjacent in the bipartite graph. Some edges are shown in the diagram in Figure 11.5.

For example,

$$\{8\heartsuit, K\spadesuit, Q\spadesuit, 2\diamondsuit, 6\diamondsuit\} \leftarrow \tag{11.4}$$

is an element of X on the left. If the audience selects this set of 5 cards, then there are many different 4-card sequences on the right in set Y that the Assistant could choose to reveal, including $(8\heartsuit, K\spadesuit, Q\spadesuit, 2\diamondsuit)$, $(K\spadesuit, 8\heartsuit, Q\spadesuit, 2\diamondsuit)$, and $(K\spadesuit, 8\heartsuit, 6\diamondsuit, Q\spadesuit)$.

What the Magician and his Assistant need to perform the trick is a *matching* for the X vertices. If they agree in advance on some matching, then when the audience selects a set of 5 cards, the Assistant reveals the matching sequence of 4 cards. The Magician uses the matching to find the audience's chosen set of 5 cards, and so he can name the one not already revealed.

For example, suppose the Assistant and Magician agree on a matching containing the two bold edges in Figure 11.5. If the audience selects the set

$$\{8\heartsuit, K\spadesuit, Q\spadesuit, 9\clubsuit, 6\diamondsuit\}, \tag{11.5}$$

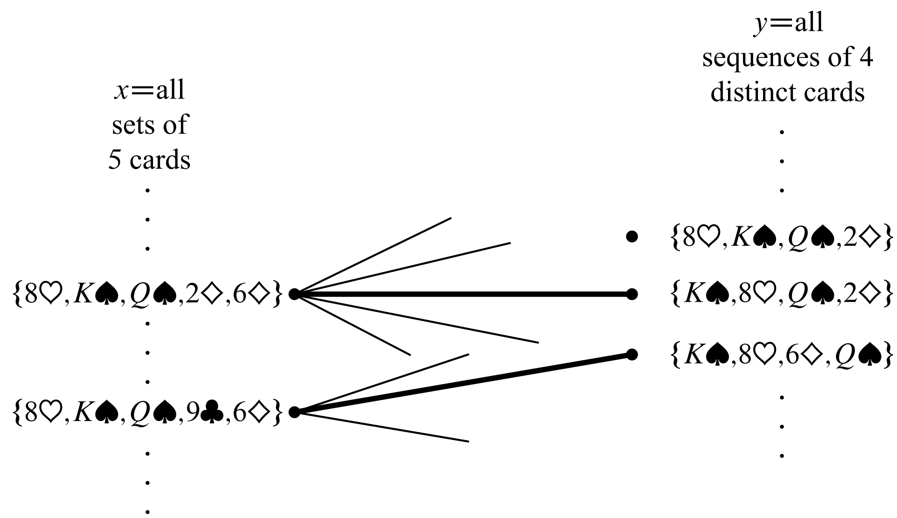


Figure 11.5 The bipartite graph where the nodes on the left correspond to *sets* of 5 cards and the nodes on the right correspond to *sequences* of 4 cards. There is an edge between a set and a sequence whenever all the cards in the sequence are contained in the set.

then the Assistant reveals the corresponding sequence

$$(K\spadesuit, 8\heartsuit, 6\diamondsuit, Q\spadesuit). \tag{11.6}$$

Using the matching, the Magician sees that the hand (11.5) is matched to the sequence (11.6), so he can name the one card in the corresponding set not already revealed, namely, the 9♣. Notice that the fact that the sets are *matched*, that is, that different sets are paired with *distinct* sequences, is essential. For example, if the audience picked the previous hand (11.4), it would be possible for the Assistant to reveal the same sequence (11.6), but he better not do that; if he did, then the Magician would have no way to tell if the remaining card was the 9♣ or the 2◇.

So how can we be sure the needed matching can be found? The answer is that each vertex on the left has degree $5 \cdot 4! = 120$, since there are five ways to select the card kept secret and there are 4! permutations of the remaining 4 cards. In addition, each vertex on the right has degree 48, since there are 48 possibilities for the fifth card. So this graph is *degree-constrained* according to Definition 5.2.6, and therefore satisfies Hall's matching condition.

In fact, this reasoning shows that the Magician could still pull off the trick if 120 cards were left instead of 48, that is, the trick would work with a deck as large as 124 different cards—without any magic!

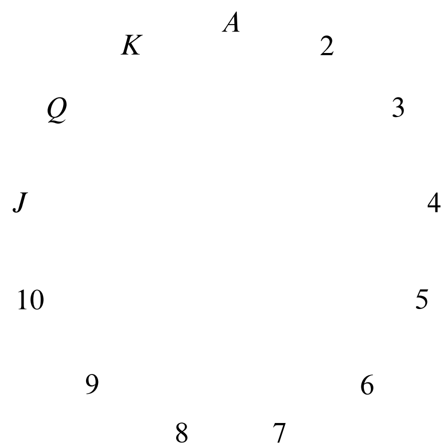


Figure 11.6 The 13 card ranks arranged in cyclic order.

11.11.2 The Real Secret

But wait a minute! It's all very well in principle to have the Magician and his Assistant agree on a matching, but how are they supposed to remember a matching with $\binom{52}{5} = 2,598,960$ edges? For the trick to work in practice, there has to be a way to match hands and card sequences mentally and on the fly.

We'll describe one approach. As a running example, suppose that the audience selects:

$$10\heartsuit \leftarrow 9\diamondsuit \leftarrow 3\heartsuit \leftarrow Q\spadesuit \leftarrow J\diamondsuit.$$

- The Assistant picks out two cards of the same suit. In the example, the assistant might choose the $3\heartsuit$ and $10\heartsuit$. This is always possible because of the Pigeonhole Principle—there are five cards and 4 suits so two cards must be in the same suit.

- The Assistant locates the ranks of these two cards on the cycle shown in Figure 11.6. For any two distinct ranks on this cycle, one is always between 1 and 6 hops clockwise from the other. For example, the $3\heartsuit$ is 6 hops clockwise from the $10\heartsuit$.

- The more counterclockwise of these two cards is revealed first, and the other becomes the secret card. Thus, in our example, the $10\heartsuit$ would be revealed, and the $3\heartsuit$ would be the secret card. Therefore:

  – The suit of the secret card is the same as the suit of the first card revealed.

  – The rank of the secret card is between 1 and 6 hops clockwise from the rank of the first card revealed.

- All that remains is to communicate a number between 1 and 6. The Magician and Assistant agree beforehand on an ordering of all the cards in the deck from smallest to largest such as:

$$A\clubsuit\ A\diamondsuit\ A\heartsuit\ A\spadesuit\ 2\clubsuit\ 2\diamondsuit\ 2\heartsuit\ 2\spadesuit\ \ldots\ K\heartsuit\ K\spadesuit$$

  The order in which the last three cards are revealed communicates the number according to the following scheme:

$$\begin{array}{lllll}
(& \text{small}, & \text{medium}, & \text{large} &) = 1 \\
(& \text{small}, & \text{large}, & \text{medium} &) = 2 \\
(& \text{medium}, & \text{small}, & \text{large} &) = 3 \\
(& \text{medium}, & \text{large}, & \text{small} &) = 4 \\
(& \text{large}, & \text{small}, & \text{medium} &) = 5 \\
(& \text{large}, & \text{medium}, & \text{small} &) = 6
\end{array}$$

  In the example, the Assistant wants to send 6 and so reveals the remaining three cards in large, medium, small order. Here is the complete sequence that the Magician sees:

$$10\heartsuit\quad Q\spadesuit\quad J\diamondsuit\quad 9\diamondsuit$$

- The Magician starts with the first card, $10\heartsuit$, and hops 6 ranks clockwise to reach $3\heartsuit$, which is the secret card!

So that's how the trick can work with a standard deck of 52 cards. On the other hand, Hall's Theorem implies that the Magician and Assistant can *in principle* perform the trick with a deck of up to 124 cards. It turns out that there is a method which they could actually learn to use with a reasonable amount of practice for a 124-card deck, but we won't explain it here.[6]

11.11.3 The Same Trick with Four Cards?

Suppose that the audience selects only *four* cards and the Assistant reveals a sequence of *three* to the Magician. Can the Magician determine the fourth card?

Let X be all the sets of four cards that the audience might select, and let Y be all the sequences of three cards that the Assistant might reveal. Now, on one hand, we have

$$|X| = \binom{52}{4} = 270,725$$

[6]See *The Best Card Trick* by Michael Kleber for more information.

by the Subset Rule. On the other hand, we have

$$|Y| = 52 \cdot 51 \cdot 50 = 132,600$$

by the Generalized Product Rule. Thus, by the Pigeonhole Principle, the Assistant must reveal the *same* sequence of three cards for at least

$$\left\lceil \frac{270,725}{132,600} \right\rceil \Bigg(= 3$$

different four-card hands. This is bad news for the Magician: if he sees that sequence of three, then there are at least three possibilities for the fourth card which he cannot distinguish. So there is no legitimate way for the Assistant to communicate exactly what the fourth card is!

11.11.4 Never Say Never

No sooner than we finished proving that the Magician can’t pull off the trick with four cards instead of five, a student showed us a way that it might be doable after all. The idea is to place the three cards on a table one at a time instead of revealing them all at once. This provides the Magician with two completely independent sequences of three cards: one for the *temporal* order in which the cards are placed on the table, and one for the *spatial* order in which they appear once placed.

For example, suppose the audience selects

$$10\heartsuit \leftarrow 9\diamondsuit \leftarrow 3\heartsuit \leftarrow Q\spadesuit \leftarrow$$

and the assistant decides to reveal

$$10\heartsuit \leftarrow 9\diamondsuit \leftarrow Q\spadesuit.$$

The assistant might decide to reveal the $Q\spadesuit$ first, the $10\heartsuit$ second, and the $9\diamondsuit$ third, thereby production the *temporal* sequence

$$(Q\spadesuit, 10\heartsuit, 9\diamondsuit).$$

If the $Q\spadesuit$ is placed in the middle position on the table, the $10\heartsuit$ is placed in the rightmost position on the table, and the $9\diamondsuit$ is placed in the leftmost position on the table, the *spatial* sequence would be

$$(9\diamondsuit, Q\spadesuit, 10\heartsuit).$$

In this version of the card trick, X consists of all sets of 4 cards and Y consists of all *pairs* of sequences of the same 3 cards. As before, we can create a bipartite graph where an edge connects a set S of 4 cards in X with a pair of sequences in Y if the 3 cards in the sequences are in S.

The degree of every node in X is then

$$4 \cdot 3! \cdot 3! = 144$$

since there are 4 choices for which card is not revealed and 3! orders for each sequence in the pair.

The degree of every node in Y is 49 since there are $52 - 3 = 49$ possible choices for the 4th card. Since $144 \geq \not< 49$, we can use Hall's Theorem to establish the existing of a matching for X.

Hence, the magic trick *is* doable with 4 cards—the assistant just has to convey more information. Can you figure out a convenient way to pull off the trick on the fly?

So what about the 3-card version? Surely that is not doable....