Solutions to In-Class Problems Week 9, Fri.

Problem 1. Solve the following problems using the Pigeonhole Principle. For each problem, try to identify the *pigeons*, the *pigeonholes*, and a *rule* assigning each pigeon to a pigeonhole.

(a) In a room of 500 people, there exist two who share a birthday.

Solution. The pigeons are the 500 people. The pigeonholes are 366 possible birthdays. Map each person to his or her own birthday. Since there 500 people and 366 birthdays, some two people must have the same birthday by the Pigeonhole Principle. ∎

(b) Every MIT ID number starts with a 9 (we think). Suppose that each of the 101 students in 6.042 sums the nine digits of his or her ID number and doubles the result. Explain why two students' results must be the same.

Solution. The students are the pigeons, the possible results are the pigeonholes, and we map each student to the result calculated from his or her MIT ID number. Every sum is in the range from 9 to $9 + 8 \cdot 9 = 81$, and since results are obtained by doubling these sums, there are also 81 possible pigeonholes. Since there are more pigeons than pigeonholes, there must be two pigeons in the same pigeonhole; in other words, there must be two students with the same result. ∎

(c) In every set of 100 integers, there exist two whose difference is a multiple of 37.

Solution. The pigeons are the 100 integers. The pigeonholes are the numbers 0 to 36. Map integer k to $k \text{ rem } 37$. Since there are 100 pigeons and only 37 pigeonholes, two pigeons must go in the same pigeonhole. This means $k_1 \text{ rem } 37 = k_2 \text{ rem } 37$, which implies that $k_1 - k_2$ is a multiple of 37. ∎

(d) For any five points inside a unit square, there are two points at distance less than $\frac{1}{\sqrt{2}}$.

Solution. The pigeons are the points. The pigeonholes are the four subsquares of the unit square, each of side length $\frac{1}{2}$. There are five pigeons and four pigeonholes, and a pigeon maps to the subsquare it is in (points on a boundary get assigned to the leftmost lowest possible subsquare) so more than one point must be in the same subsquare. The points in the same subsquare are at distance less than $1/\sqrt{2}$, because the most distant points in the quadrant are at opposite corners at exactly this distance, but only one of these corners can be *inside* the square (not on the boundary), so the distance between the points in the quadrant must actually be less than $1/\sqrt{2}$. ■

(e) For any five points inside an equilateral triangle of side length 2, there are two points at distance less than 1.

Solution. The pigeons are the points. The pigeonholes are the four sub-equilateral triangles of side length 1. There are five pigeons and four pigeonholes, so more than one point must be in the same sub-equilateral triangle. Points inside the same sub-equilateral triangle are at distance less than 1. ■

(f) Prove that every finite undirected graph with two or more vertices has two vertices of the same degree.

Solution. Suppose a graph has $n \geq 2$ vertices. Hence, the vertex degrees must be between 0 and $n-1$, inclusive. However, both 0 and $n-1$ cannot both occur: if there is an isolated (degree 0) vertex, then the remaining vertices can have degree at most $n-2$. So in no graph are there ever more than $n-1$ possible degree values. Since there are n vertices, and at most $n-1$ degrees, the Pigeonhole Principle implies there must be two vertices with the same degree. ■

Problem 2. Your 6.001 tutorial has 12 students, who are supposed to break up into 4 groups of 3 students each. Your TA has observed that the students waste too much time trying to form balanced groups, so he decided to pre-assign students to groups and email the group assignments to his students.

(a) Your TA has a list of the 12 students in front of him, so he divides the list into consecutive groups of 3. For example, if the list is ABCDEFGHIJKL, the TA would define a sequence of four groups to be $(\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\})$. This way of forming groups defines a mapping from a list of twelve students to a sequence of four groups. This is a k-to-1 mapping for what k?

Solution. Two lists map to the same sequence of groups iff the first 3 students are the same on both lists, and likewise for the second, third, and fourth consecutive sublists of 3 students. So for a given sequence of 4 groups, the number of lists which map to it is

$$(3!)^4$$

because there are 3! ways to order the students in each of the 4 consecutive sublists. ■

(b) A group assignment specifies which students are in the same group, but not any order in which the groups should be listed. If we map a sequence of 4 groups,

$$(\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\}),$$

into a group assignment

$$\{\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\}\},$$

this mapping is j-to-1 for what j?

Solution. $4!$. ■

(c) How many group assignments are possible?

Solution.

$$\frac{12!}{4! \cdot (3!)^4} = 15400$$

different assignments.

There are $12!$ possible lists of students, and we can map each list to an assignment by first mapping the list to a sequence of four groups, and then mapping the sequence to the assignment. Since the first map is $(3!)^4$-to-1 and and the second is $4!$-to-1, the composite map is $(3!)^4 \cdot 4!$-to-1. So by the Division Rule, $12! = \left((3!)^4 \cdot 4!\right) A$ where A is the number of assignments. ■

(d) In how many ways can $3n$ students be broken up into n groups of 3?

Solution.

$$\frac{(3n)!}{(3!)^n n!}.$$

This follows simply but replacing "12" by "3n" in the solution to the previous problem parts. ■

Problem 3. Answer the following quesions using the Generalized Product Rule.

(a) Next week, I'm going to get really fit! On day 1, I'll exercise for 5 minutes. On each subsequent day, I'll exercise 0, 1, 2, or 3 minutes more than the previous day. For example, the number of minutes that I exercise on the seven days of next week might be 5, 6, 9, 9, 9, 11, 12. How many such sequences are possible?

Solution. The number of minutes on the first day can be selected in 1 way. The number of minutes on each subsequent day can be selected in 4 ways. Therefore, the number of exercise sequences is $1 \cdot 4^6$ by the extended product rule. ■

(b) An r-***permutation*** of a set is a sequence of r distinct elements of that set. For example, here are all the 2-permutations of $\{a, b, c, d\}$:

$$\begin{array}{ccc} (a,b) & (a,c) & (a,d) \\ (b,a) & (b,c) & (b,d) \\ (c,a) & (c,b) & (c,d) \\ (d,a) & (d,b) & (d,c) \end{array}$$

How many r-permutations of an n-element set are there? Express your answer using factorial notation.

Solution. There are n ways to choose the first element, $n-1$ ways to choose the second, $n-2$ ways to choose the third, . . . , and $n-r+1$ ways to choose the r-th element. Thus, there are:

$$n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) = \frac{n!}{(n-r)!}$$

r-permutations of an n-element set. ■

(c) How many $n \times n$ matrices are there with *distinct* entries drawn from $\{1, \ldots, p\}$, where $p \geq n^2$?

Solution. There are p ways to choose the first entry, $p-1$ ways to choose the second for each way of choosing the first, $p-2$ ways of choosing the third, and so forth. In all there are

$$p(p-1)(p-2)\cdots(p-n^2+1) = \frac{p!}{(p-n^2)!}$$

such matrices. Alternatively, this is the number of n^2-permutations of a p element set, which is $p!/(p-n^2)!$. ■

Problem 4. A certain company wants to have security for their computer systems. So they have given everyone a name and password. A length 10 word containing each of the characters:

a, d, e, f, i, l, o, p, r, s,

is called a *cword*. A password will be a cword which does not contain any of the subwords ”fails”, ”failed”, or ”drop”.

For example, the following two words are passwords:

adefiloprs, srpolifeda,

but the following three cwords are not:

a**drop**eflrs, **failedrop**s, **drop**e**fails**.

(a) How many cwords contain the subword "drop"?

Solution. Such cwords are obtainable by taking the word "drop" and the remaining 6 letters in any order. There are 7! permutations of these 7 items. ■

(b) How many cwords contain both "drop" and "fails"?

Solution. Take the words "drop" and "fails" and the remaining letter "e" in any order. So there are 3! such cwords. ■

(c) Use the Inclusion-exclusion Principle to find a simple formula for the number of passwords.

Solution. There are $7!$ cwords that contain "drop", $6!$ that contain "fails", and $5!$ that contain "failed". There are $3!$ cwords containing both "drop" and "fails". No cword can contain both "fails" and "failed". The cwords containing both "drop" and "failed" come from taking the subword "failedrop" and the remaining letter "s" in any order, so there are $2!$ of them. So by Inclusion-exclusion, we have the number of cwords containing at least one of the three forbidden subwords is

$$(7! + 6! + 5!) - (3! + 0 + 2!) + 0 = 5!(48) - 8.$$

Among the $10!$ cwords, the remaining ones are passwords, so the number of passwords is

$$10! - 5!(48) + 8 = 3,623,048.$$

■