Solutions to In-Class Problems Week 8, Wed.

Problem 1. We begin with two large glasses. The first glass contains a pint of water, and the second contains a pint of wine. We pour 1/3 of a pint from the first glass into the second, stir up the wine/water mixture in the second glass, and then pour 1/3 of a pint of the mix back into the first glass and repeat this pouring back-and-forth process a total of n times.

(a) Describe a closed form formula for the amount of wine in the first glass after n back-and-forth pourings.

Solution. The state of the system of glasses/wine/water at the beginning of a round of pouring and pouring back is determined by the total amount of wine in the first glass. Suppose at the beginning of some round, the first glass contains w pints of wine, $0 \leq w \leq 1$ and $1 - w$ pints of water. The second glass contains the rest of the wine and water.

Pouring 1/3 pint from first glass to second leaves 2/3 pints of liquid and $(2/3)w$ wine in the first glass, and 4/3 pints of liquid and $1 - (2/3)w$ wine in the second glass. Pouring 1/3 pint back from second into first transfers a proportion of (1/3)/(4/3) of the wine in the second glass into the first. So the round completes with both glasses containing a pint of liquid, and the first glass containing

$$(2/3)w + (1/4)(1 - (2/3)w) = 1/4 + w/2$$

pints of wine. After one more round, the first glass contains

$$1/4 + (1/4 + w/2)/2 = 1/4 + 1/8 + w/2^2$$

pints of wine, and after n more rounds

$$\begin{aligned} w/2^n + \Sigma_{i=1}^{n}(1/2)^{i+1} &= w/2^n + (1/2)\Sigma_{i=1}^{n}(1/2)^i \\ &= w/2^n + (1/2)(-1 + \Sigma_{i=0}^{n}(1/2)^i) \\ &= w/2^n + (1/2)(-1 + (1 - (1/2)^{n+1})/(1 - 1/2)) \\ &= w/2^n - 1/2 + 1 - (1/2)^{n+1} \\ &= w/2^n + 1/2 - (1/2)^{n+1}. \end{aligned}$$

Since $w = 0$ initially, the pints of wine in the first glass after n rounds is

$$1/2 - (1/2)^{n+1}.$$

■

(b) What is the limit of the amount of wine in each glass as n approaches infinity?

Solution. The limiting amount of wine in the first glass approaches 1/2 from below as n approaches infinity. In fact, it approaches 1/2 no matter how the wine was initially distributed. This of course is what you would expect: after a thorough mixing the glasses should contain essentially the same amount of wine. ■

Problem 2. Suppose you were about to enter college today and a college loan officer offered you the following deal: \$25,000 at the start of each year for four years to pay for your college tuition and an option of choosing one of the following repayment plans:

Plan A: Wait four years, then repay \$20,000 at the start of each year for the next ten years.

Plan B: Wait five years, then repay \$30,000 at the start of each year for the next five years.

Suppose the annual interest rate paid by banks is 7% and does not change in the future.

(a) Assuming that it's no hardship for you to meet the terms of either payback plan, which one is a better deal? (You will need a calculator.)

Solution. $\$1$ today will be worth $\$1.07$ next year, and $\$1.07^2$ the year after, *etc*. So set $r = \frac{1}{1.07}$. Then:

$$\begin{aligned}
&\text{current value of Plan A} \\
&= \sum_{y=4}^{13} 20000 \cdot r^y \\
&= \sum_{y=0}^{9} 20000 \cdot r^{y+4} \\
&= r^4 \cdot \sum_{y=0}^{9} 20000 \cdot r^y \\
&= 20000r^4 \cdot \sum_{y=0}^{9} r^y \\
&= 20000r^4 \cdot \frac{1 - r^{10}}{1 - r} \\
&= \$114,66.69
\end{aligned}$$

$$\begin{aligned}
&\text{current value of Plan B} \\
&= \sum_{y=5}^{9} 30000 \cdot r^y \\
&= \sum_{y=0}^{4} 30000 \cdot r^{y+5} \\
&= r^5 \cdot \sum_{y=0}^{4} 30000 \cdot r^y \\
&= 30000r^5 \cdot \sum_{y=0}^{4} r^y \\
&= 30000r^5 \cdot \frac{1 - r^5}{1 - r} \\
&= \$93,840.63
\end{aligned}$$

You should clearly take Plan B. You will be paying back much less in today's dollars. ■

(b) What is the loan officer's effective profit (in today's dollars) on the loan?

Solution. The value of the money you are given is:

$$\begin{aligned}
Loan &= \sum_{y=0}^{3} 25000 \cdot r^y \\
&= 25000 \cdot \sum_{y=0}^{3} r^y \\
&= 25000 \cdot \frac{1 - r^4}{1 - r} \\
&= \$90,607.90
\end{aligned}$$

Therefore, the loan officer's profit is effectively $\$3,233$. (Or $\$24,059$ if we are not on the ball). ■

Problem 3. Riemann's Zeta Function $\zeta(k)$ is defined to be the infinite summation:

$$1 + \frac{1}{2^k} + \frac{1}{3^k} \cdots = \sum_{j \geq 1} \frac{1}{j^k}$$

Below is a proof that

$$\sum_{k \geq 2} (\zeta(k) - 1) = 1$$

Justify each line of the proof. (P.S. The purpose of this exercise is to highlight some of the rules for manipulating series. Don't worry about the significance of this identity.)

$$\begin{aligned}
\sum_{k\geq 2}(\zeta(k)-1) &= \sum_{k\geq 2}\left[\left(\sum_{j\geq 1}\frac{1}{j^k}\right)-1\right] && (1)\\
&= \sum_{k\geq 2}\sum_{j\geq 2}\frac{1}{j^k} && (2)\\
&= \sum_{j\geq 2}\sum_{k\geq 2}\frac{1}{j^k} && (3)\\
&= \sum_{j\geq 2}\frac{1}{j^2}\sum_{k\geq 0}\frac{1}{j^k} && (4)\\
&= \sum_{j\geq 2}\frac{1}{j^2}\cdot\frac{1}{1-1/j} && (5)\\
&= \sum_{j\geq 2}\frac{1}{j(j-1)} && (6)\\
&= \lim_{n\to\infty}\sum_{j=2}^{n}\frac{1}{j(j-1)} && (7)\\
&= \lim_{n\to\infty}\sum_{j=2}^{n}\frac{1}{j-1}-\frac{1}{j} && (8)\\
&= \lim_{n\to\infty}(1-\frac{1}{n}) && (9)\\
&= 1 && (10)
\end{aligned}$$

Solution. (1) Definition of $\zeta(k)$.

(2) Because $\sum_{j\geq 1}\frac{1}{j^k} = 1+\sum_{j\geq 2}\frac{1}{j^k}$.

(3) Reordering; this is ok because at this point all terms are positive.

(4) Because $\sum_{k\geq 2}\frac{1}{j^k} = \sum_{k\geq 0}\frac{1}{j^{k+2}} = \sum_{k\geq 0}\frac{1}{j^k\cdot j^2} = \frac{1}{j^2}\sum_{k\geq 0}\frac{1}{j^k}$.

(5) Sum of a geometric series.

(6) Algebra inside every summand.

(7) Definition of infinite summation.

(8) Algebra inside every summand.

(9) The sum telescopes: 1 is added once; every one of $\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n-1}$ is subtracted once and then added once; $\frac{1}{n}$ is subtracted once.

(10) Simple limits. ■