In-Class Problems Week 10, Mon.

Problem 1. A bipartite graph is *regular* if every vertex on the left has the same degree, c, and every vertex on the right has the same degree, d.

(a) Prove the following:

Corollary. *A regular bipartite graph has a matching for the vertices on the left iff* $c \geq d > 0$.

Hint: Consider the set of edges between any set, L, on the left and its set of neighbors, $N(L)$, on the right.

(b) Conclude that the Magician could pull off the Card Trick with a deck of 124 cards.

Problem 2. We have just demonstrated how to determine the 5th card in a poker hand when a collaborator reveals the other 4 cards. Describe a similar method for determining 2 hidden cards in a hand of 9 cards when your collaborator reveals the other 7 cards.

Problem 3. The Tao of BOOKKEEPER: we seek enlightenment through contemplation of the word $BOOKKEEPER$.

(a) In how many ways can you arrange the letters in the word $POKE$?

(b) In how many ways can you arrange the letters in the word BO_1O_2K? Observe that we have subscripted the O's to make them distinct symbols.

(c) Suppose we map arrangements of the letters in BO_1O_2K to arrangements of the letters in $BOOK$ by erasing the subscripts. Indicate with arrows how the arrangements on the left are mapped to the arrangements on the right.

O_2BO_1K	
KO_2BO_1	$BOOK$
O_1BO_2K	$OBOK$
KO_1BO_2	$KOBO$
BO_1O_2K	$\ldots$
BO_2O_1K	
$\ldots$	

(d) What kind of mapping is this, young grasshopper?

(e) In light of the Division Rule, how many arrangements are there of $BOOK$?

(f) Very good, young master! How many arrangements are there of the letters in $KE_1E_2PE_3R$?

(g) Suppose we map each arrangement of $KE_1E_2PE_3R$ to an arrangement of $KEEPER$ by erasing subscripts. List all the different arrangements of $KE_1E_2PE_3R$ that are mapped to $REPEEK$ in this way.

(h) What kind of mapping is this?

(i) So how many arrangements are there of the letters in $KEEPER$?

(j) *Now you are ready to face the BOOKKEEPER!*
How many arrangements of $BO_1O_2K_1K_2E_1E_2PE_3R$ are there?

(k) How many arrangements of $BOOK_1K_2E_1E_2PE_3R$ are there?

(l) How many arrangements of $BOOKKE_1E_2PE_3R$ are there?

(m) How many arrangements of $BOOKKEEPER$ are there?

(n) How many arrangements of $VOODOODOLL$ are there?

(o) **(IMPORTANT)** How many n-bit sequences contain k zeros and $(n-k)$ ones?

Remember well what you have learned: subscripts on, subscripts off.

This is the Tao of Bookkeeper.

Problem 4. Solve the following counting problems. Define an appropriate mapping (bijective or k-to-1) between a set whose size you know and the set in question.

(a) How many different ways are there to select a dozen donuts if four varieties are available?

(b) How many paths are there from $(0, 0)$ to $(10, 20)$ consisting of right-steps (which increment the first coordinate) and up-steps (which increment the second coordinate)?

(c) An independent living group is hosting nine new candidates for membership. Each candidate must be assigned a task: 1 must wash pots, 2 must clean the kitchen, 3 must clean the bathrooms, 1 must clean the common area, and 2 must serve dinner. In how many ways this be done?

(d) In how many ways can Mr. and Mrs. Grumperson distribute 13 identical pieces of coal to their two— no, three!— children for Christmas?

(e) How many solutions over the natural numbers are there to the equation:

$$x_1 + x_2 + \ldots + x_{10} \quad \leq \quad 100$$

(f) (Quiz 2, Fall '03) Suppose that two identical 52-card decks are mixed together. In how many ways can the cards in this double-size deck be arranged?