Quiz 1

Your name:____

Circle the name of your Tutorial Instructor:

David Hanson Jelani Sayan

- This quiz is **closed book** except for a personal one-page crib sheet. You may assume any of the results presented in class or in the lecture notes. Total time is 80 minutes.
- There are seven (7) problems totaling 100 points. Problems are labeled with their point values.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	15		
2	10		
3	20		
4	20		
5	10		
6	15		
7	10		
Total	100		

Problem 1 (15 points). Let G be an undirected graph. Let $P(x, y)$ mean that there is a path from vertex x to vertex y. Express each of the following sentences in terms of P, quantifiers, logical connectives, and equality, using variables that range over the vertices of G. (Reminder: there is a zero-length path from any vertex to itself.)

(a) (3 points) Vertices x and y are in the same connected component.

(b) (3 points) G has a vertex of degree zero. (Reminder: undirected graphs only have edges between distinct vertices, that is, no self-loops.)

(c) (4 points) G has at least three connected components.

(d) (5 points) There is a positive-length *simple* path from x to y.

Problem 2 (10 points). Classify each of the following binary relations as

E: An equivalence relation.

T: A Total order,

P: A Partial order that is not total.

S: A Symmetric relation that is not transitive.

N: None of the above.

(a) (2 points) The relation between times during a single day: x and y are at most twenty minutes apart. ____

(b) (2 points) The relation between times during a single day: x is more than twenty minutes later than y. ____

(c) (2 points) The relation between vertices in an arbitrary digraph: there is a path from v to w. ____

(d) (2 points) The relation between vertices in an undirected graph: there is a path from v to w. ____

(e) (2 points) The relation between Fall '05 6.042 students: student s is older but also shorter than t. ____

Problem 3 (20 points). Let $G_0 = 1$, $G_1 = 2$, $G_2 = 4$, and define

$$G_n ::= G_{n-1} + 2G_{n-2} + G_{n-3} \tag{1}$$

for $n \geq 3$. Show by induction that $G_n \leq (2.2)^n$ for all $n \geq 0$.

Problem 4 (20 points). An *intersection graph* is an undirected graph whose vertices are sets and whose edges are specified by the rule that there is an edge between vertices A and B iff $A \neq B$ and $A \cap B \neq \emptyset$.

(a) **(1 point)** Draw the intersection graph whose vertices are the sets

$$\{1,2,3\}, \{1,9,10\}, \{2,4,6,8,10\}, \{3,4,5\}, \{5,6,7\}, \{7,8,9\}$$

(b) **(3 points)** What is the chromatic number of the graph in part (a)? ____

(c) **(3 points)** What is the largest k such that the graph in part (a) is k-connected? ____

We now consider an arbitrary undirected graph, G. For any vertex, v, of G, let $I(v)$ be the set of edges incident to v.

(d) **(3 points)** Explain how to uniquely determine the vertex v given any two edges in $I(v)$.

(e) (10 points) An *incidence-set* is the set of edges incident to some vertex, that is, a set equal to $I(v)$ for some vertex v of G. Prove that if G is a graph whose vertices all have degree greater than 1, then the function, I, is an isomorphism between G and the intersection graph whose vertices are the incidence-sets of G.

Problem 5 (10 points). Two banks only allow transactions that are multiples of 3^9 dollars or 5^7 dollars. Is there a series of transactions whose net result is a payment of 1 dollar from the first bank to the second bank? Briefly explain why or why not.

Problem 6 (15 points). Each year, Santa's reindeer hold "Reindeer Games", from which Rudolph is pointedly excluded. The Games consist of a sequence of matches, where one reindeer competes against another. Draws are not possible.

On Christmas Eve, Santa produces a rank list of all his reindeer. If reindeer p lost a match to reindeer q, then p appears below q in Santa's ranking, but if he has any choice because of unplayed matches, Santa can give higher rank to the reindeer he likes better. To prevent confusion, two reindeer may not play a match if either outcome would lead to a cycle of reindeer, where each lost to the next.

Though it is only October, the 2004 Reindeer Games have already begun. We can describe the results so far with a binary relation, L, on the set of reindeer, where pLq means that reindeer p lost a match to reindeer q. Let L^+ be the corresponding positive-length path relation[1]. Note that L^+ is a partial order, so we can regard a match loser as "smaller" than the winner.

On the following page you'll find a list of terms and a sequence of statements. Add the appropriate term to each statement.

[1]Thus, reindeer p is related to reindeer q by L^+ if p lost to q or if p lost to a reindeer who lost to q or if p lost to a reindeer who lost to a reindeer who lost to q, etc.

Terms

a strict partial order	a weak partial order	a total order
comparable elements	incomparable elements	a chain
an antichain	a maximal antichain	a topological sort
a minimum element	a minimal element	
a maximum element	a maximal element	

Statements

(a) (1 point) An unbeaten reindeer is

______________________________________ of the partial order L^+.

(b) (1 point) A reindeer who has lost every match so far is

______________________________________ of the partial order L^+.

(c) (1 point) Two reindeer can *not* play a match if they are

______________________________________ of L^+.

(d) (1 point) A reindeer assured of first place in Santa's ranking is

______________________________________ of L^+.

(e) (1 point) A sequence of reindeer which *must* appear in the same order in Santa's rank list is

______________________________________.

(f) (2 points) A set of reindeer such that any two could still play a match is

______________________________________.

(g) (2 points) The fact that no reindeer loses a match to himself implies that L^+ is

______________________________________.

(h) (2 points) Santa's final ranking of his reindeer on Christmas Eve must be

______________________________________ of L^+.

(i) (2 points) No more matches are possible if and only if L^+ is

______________________________________.

(j) (2 points) Suppose that Santa has 11 reindeer. If no more matches can be played, what is the smallest possible number of matches already played? ____

Problem 7 (10 points). A *map* is a connected planar graph with a planar drawing whose face boundaries are simple cycles.

(a) (7 points) Prove that if a map has no simple cycle of length 3, then

$$e \leq 2v - 4, \tag{2}$$

where v is the number of vertices and e is the number of edges in the graph.

(b) (3 points) Prove that $K_{3,3}$ is not a map. ($K_{3,3}$ is the graph with six vertices and an edge from each of the first three vertices to each of the last three.)