Introduction to Numerical Simulation (Fall 2003)
Problem Set #7 - due November 14

1)In this problem you will examine local truncation error, stability and convergence for an example method that does not converge very well. Consider using the following integration method to solve for the $x(t)$ which satisfies $\frac{d}{dt}x(t) = \lambda x(t)$,

$$\frac{x[m+1] - x[m-1]}{2h} = \lambda x[m],$$

where $x[0] = 1$. Here $x[m]$ approximates $x(t)$ at time point $t = mh$.

a) Determine the local truncation error of this "leap-frog" method.

b) Is the method stable? Will the method converge?

c) Plot and compare the computed and the exact solution for the case $\lambda = -1$, and on the interval $t \in [0, 10]$. Use $h = 2$, $h = 0.5$, and $h = 0.1$.

d) Look carefully at your plots, and explain your results in part c.

2) In this problem you will gain practice with the techniques for estimating local truncation error, and also examine what happens to the accuracy of an integration method when there are jumps in the problem description's nonlinear function. Consider using the following integration method to solve for the $x(t)$ which satisfies $\frac{d}{dt}x(t) = f(x(t))$,

$$x[m+1] = x[m] + h\left(a_1 f(x[m]) + a_2 f(x[m] + 2/3hf(x[m]))\right)$$

where $x[0] = 1$. Note that $x[m]$ is intended to approximate $x(t)$ at time point $t = mh$.

a) Determine the values of a_1 and a_2 so that the local truncation error of the above method is order h^3. One approach for computing the coefficients is to test the formula using $x(t) = 1, t, t^2, etc.$

b) Suppose $f(x(t)) = -0.5x(t)$. Use the above method, with the coefficients you determined in (a), to compute the solution for the interval $t \in [0, 1]$. Compare your computed solution to the exact solution at $t = 1$ for $h = 1/4, 1/8, 1/16, 1/32, ..., 2^{-10}$. How fast is the above method converging to the exact solution?

c) Now suppose $f(x(t)) = -x(t)$ for $x(t) > 0.5$ and $f(x(t)) = -0.5x(t)$ otherwise. Use the above method, with the coefficients you determined in (a), to compute the solution for the interval $t \in [0, 1]$. Compare your computed solution to the exact solution at $t = 1$ $(0.5e^{-0.5(1+log(0.5))})$

for $h = 1/4, 1/8, 1/16, 1/32, ..., 2^{-10}$. How fast is the above method converging to the exact solution? Explain your results.

3) In this problem you will compare the finite-difference and shooting methods for solving a non-linear problem. You should see that the two methods are quite different. Consider the scalar equation

$$\frac{d}{dt}x(t) = \sinh \alpha x + \cos 2\pi t$$

Suppose this equation has a periodic steady-state solution of period $T = 1$, i.e., $x(t) = x(t + nT)$ where n is an integer. In this problem you will examine two different approaches for finding this periodic solution. That is, you will use two different techniques to find a solution to the above equation, on the interval $t \in [0, 1]$, which also satisfies the constraint that $x(0) = x(1)$.

a) Set $\alpha = -1$, and use the finite-difference method to solve the above periodic problem. Use backward-euler for the time-discretization, $\Delta t = 0.01$ so there will be 100 nodes in your finite-difference discretization, and an initial guess $x(t) = 1$.

b) Again using backward-euler for the time-discretization, use the shooting method to obtain $x(t)$ for $\alpha = -1$. Use the initial guess $x(0) = 1$.

c) Now let $\alpha = 10$. Using the same discretization and initial guess as in part (a), solve for $x(t)$ using the finite-difference method. What happens when you try to use the shooting method on this problem?

d) Try using the value of $x(0)$ found by the finite-difference method as an initial guess for the shooting method. How much can you perturb $x(0)$ and still have the shooting method converge?