Transient Analysis of First Order RC and RL circuits

The circuit shown on Figure 1 with the switch open is characterized by a particular operating condition.
Since the switch is open, no current flows in the circuit ($i=0$) and $vR=0$. The voltage across the capacitor, vc, is not known and must be defined. It could be that $vc=0$ or that the capacitor has been charged to a certain voltage $vc = V_0$.
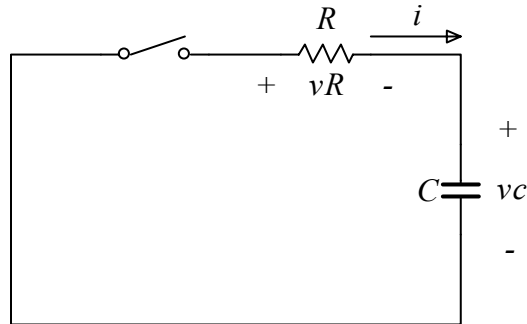

Figure 1

Let us assume the non-trivial **initial equilibrium or initial steady state condition** for the capacitor voltage $vc = V_0$ and let's close the switch at time $t = 0$, resulting in the circuit shown on Figure 2.
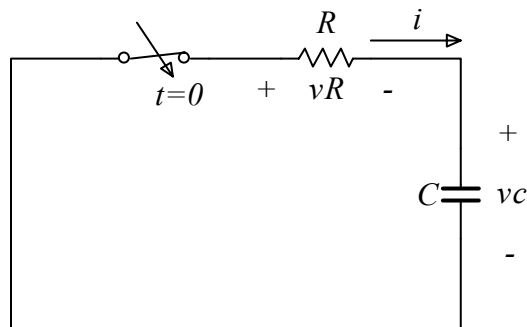

Figure 2

After closing the switch, current will begin to flow in the circuit. Energy will be dissipated in the resistor and eventually all energy initially stored in the capacitor,

$E_C = \dfrac{1}{2} C \, vc^2$, will be dissipated as heat in the resistor. After a long time, the current will be zero and the circuit will reach a new, albeit trivial, equilibrium or steady state condition ($i=0$, $vc=0$, $vR=0$).

The transient characteristics of the circuit describes the behavior of the circuit during the transition from one steady state condition to another. In this class we will develop the tools for describing and understanding this transient phenomena.

Source Free RC Circuit

As out first example let's consider the source free RC circuit shown on Figure 3.
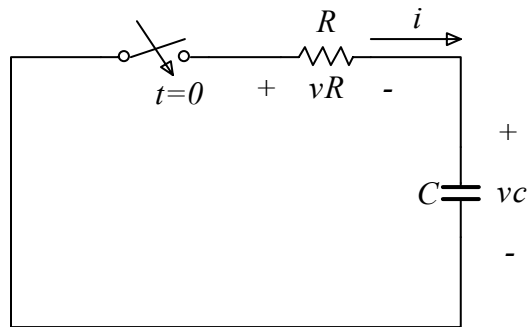

Figure 3

Let's assume that initially the "ideal" capacitor is charged with a voltage $vc_{t=0^-} = V_0$.

At time $t = 0$, the switch is closed, current begins to flow in the circuit and we would like to obtain the form of the voltage vc as a function of time for $t>0$. Since the voltage across the capacitor must be continuous the voltage at $t = 0^+$ is also Vo.

Our first task is to determine the equation that describes the behavior of this circuit. This is accomplished by using Kirchhoff's laws. Here we use KLV which gives,

$$v_R(t) + v_c(t) = 0 \qquad (0.1)$$

Using the current voltage relationship of the resistor and the capacitor, Equation (0.1) becomes

$$RC \frac{dv_c(t)}{dt} + v_c(t) = 0 \qquad (0.2)$$

Note that the product RC has the unit of time. (Ohm)(Farad) $\rightarrow$ seconds
RC is called the time constant of the circuit and it is often assigned the variable $\tau = RC$.

Equation (0.2) along with the initial condition, $vc_{t=0} = V_0$ describe the behavior of the circuit for $t>0$. In fact, since the circuit is not driven by any source the behavior is also called the **natural response** of the circuit.

Equation (0.2) is a first order homogeneous differential equation and its solution may be easily determined by separating the variables and integrating. However we will employ a more general approach that will also help us to solve the equations of more complicated circuits later on.

Assume that a solution to Equation (0.2) is of the form given by

$$vc(t) = Ae^{st} \qquad (0.3)$$

The parameters A and s are to be determined by the specific characteristics of the system. By substituting equation (0.3) into equation (0.2) we obtain,

$$RCAse^{st} + Ae^{st} = 0 \qquad (0.4)$$

Or equivalently,

$$(RC\ s + 1)Ae^{st} = 0 \qquad (0.5)$$

The only non-trivial solution of Equation (0.5) follows from

$$(RC\ s + 1) = 0 \qquad (0.6)$$

This is called the characteristic equation of the system. Therefore s is

$$s = -\frac{1}{RC} \qquad (0.7)$$

And the solution is

$$vc(t) = Ae^{\frac{-t}{RC}} = Ae^{\frac{-t}{\tau}} \qquad (0.8)$$

The constant A may now be determined by applying the initial condition $vc_{t=0} = V_0$ which gives

$$A = V_0 \qquad (0.9)$$

And the final solution is

$$vc(t) = V_0 e^{\frac{-t}{RC}} \qquad (0.10)$$

The plot of the voltage vc is shown on Figure 4.
At $t=0$ the voltage starts at V_0 and subsequently it exponentially decays to zero.
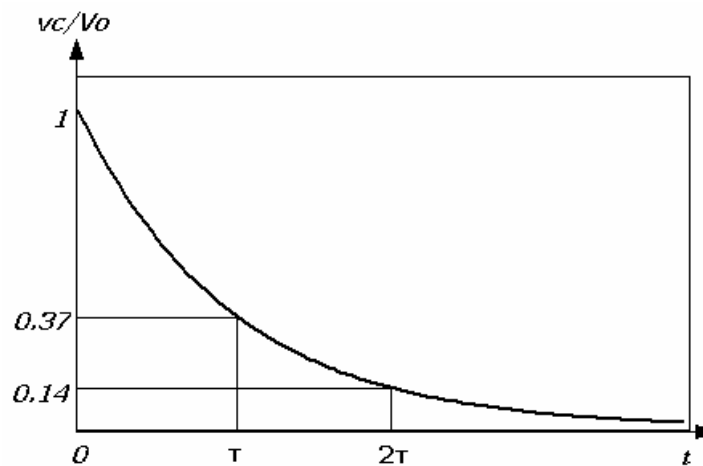

Figure 4

Source Free RL Circuit

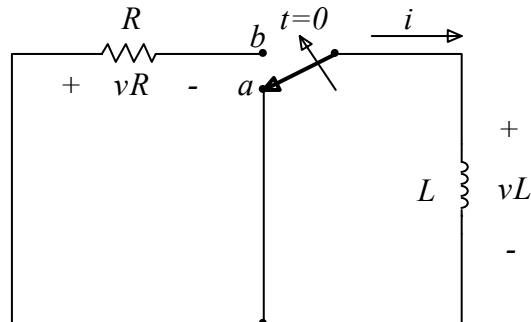Now let's consider the RL circuit shown on Figure 5.

Figure 5

Initially the switch is at position *a* and there is a current I_0 circulating in the loop containing the "ideal" inductor. This is the initial equilibrium state of the circuit and its schematic is shown on Figure 6(a). At time t=0 the switch is moved from position *a* to position *b*. Now the resistor *R* is incorporated in the circuit and the current I_0 begins to flow through it as shown Figure 6(b).

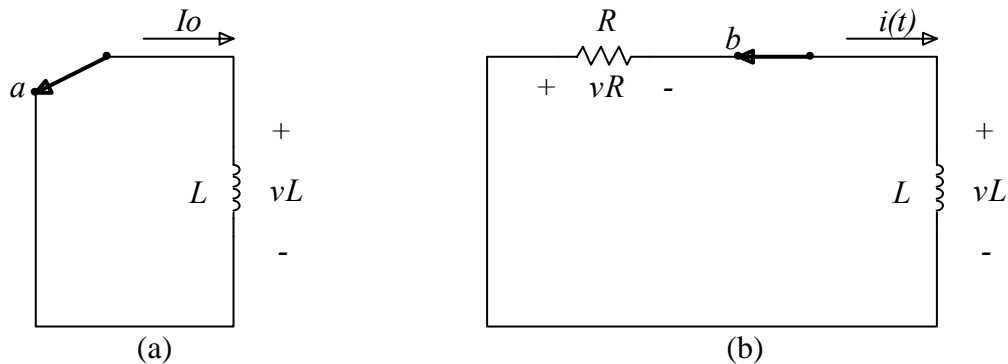(a)   (b)

Figure 6

Our goal is to determine the form of the current *i(t)*.

We start by deriving the equation that describes the behavior of the circuit for *t>0*. KVL around the mesh of the circuit on Figure 6(b) gives.

$$v_R(t) + v_L(t) = 0 \tag{0.11}$$

Using the current voltage relationship of the resistor and the inductor, Equation (0.11) becomes

$$\frac{L}{R}\frac{di(t)}{dt} + i(t) = 0 \tag{0.12}$$

The ratio $\dfrac{L}{R}$ has the units of time as can be seen by simple dimensional analysis.

By assuming a solution of the form,

$$i(t) = Be^{st} \tag{0.13}$$

Equation (0.12) becomes after substitution

$$\left(\frac{L}{R}s + 1\right)Be^{st} = 0 \tag{0.14}$$

The non-trivial solution of Equation (0.14) is

$$s = -\frac{R}{L} \tag{0.15}$$

And the solution becomes

$$i(t) = Be^{\frac{-t}{L/R}} \tag{0.16}$$

The constant B may now be determined by considering the initial condition of the circuit $i_{t=0} = I_0$, which gives $B = I_0$. And the completed solution is

$$i(t) = I_0 e^{\frac{-t}{L/R}} \tag{0.17}$$

The ratio $\dfrac{L}{R}$ is the characteristic time constant of the RL circuit. Figure 7 shows the normalized plot of *i(t)*.
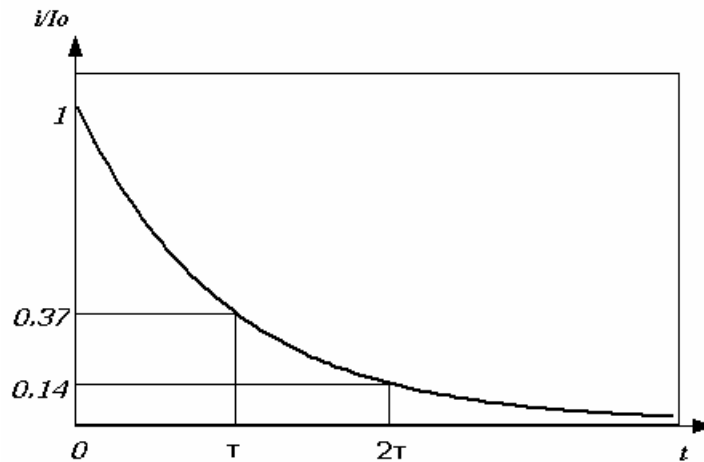

Figure 7

RC and RL circuits with multiple resistors.

The capacitor of the circuit on Figure 8 is initially charged to a voltage *Vo*. At time t=0 the switch is closed and current flows in the circuit. The capacitor sees a Thevenin equivalent resistance which is

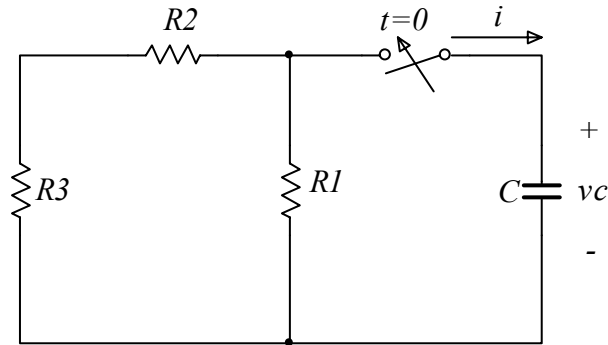$$R_{eq} = \frac{(R2+R3)R1}{R1+R2+R3} \tag{0.18}$$

Figure 8

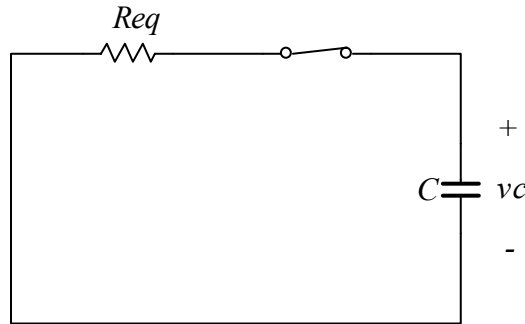Therefore once the switch is closed, the equivalent circuit becomes

Figure 9

The characteristic time is now given by

$$\tau = R_{eq}C \tag{0.19}$$

And the evolution of the voltage *vc* is

$$vc(t) = V_0 e^{\frac{-t}{R_{eq}C}} \tag{0.20}$$

RL Circuit with multiple resistors and inductors.

Let's consider the circuit shown on Figure 10 which contains multiple inductors and resistors. Initially the switch is closed and has been closed for a long time. At time $t=0$ the switch opens and we would like to obtain the transient behavior of the circuit for $t>0$. In particular we are interested in determining the current $i(t)$ as indicated on the circuit of Figure 11
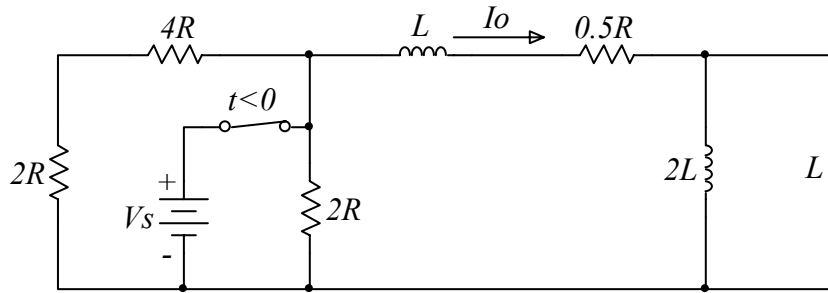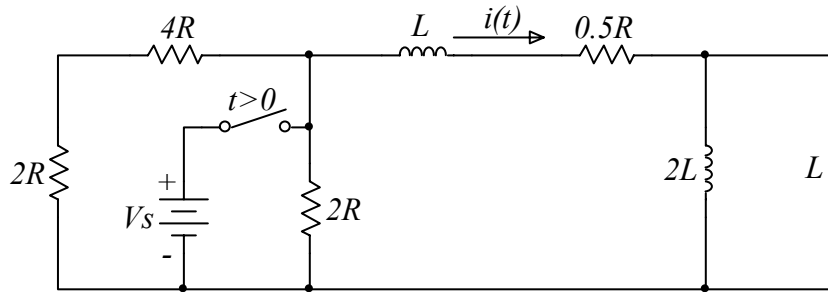

Figure 10

Figure 11

In order to find the initial ($t = 0^+$) current flowing in the circuit we consider the circuit on Figure 10. The circuit may be simplified by combining the resistors and taking into account the operational characteristics of the inductor at equilibrium. Since under DC conditions the inductors act as short circuits the corresponding circuit becomes
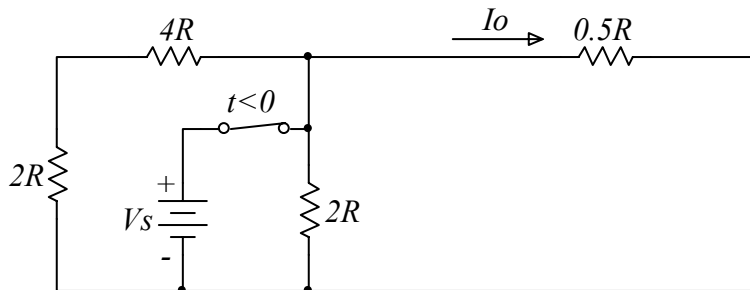

Figure 12

And thus the current $Io = 2\dfrac{Vs}{R}$. Therefore at the moment that the switch is opened, the current is known. This is the initial condition for our problem.

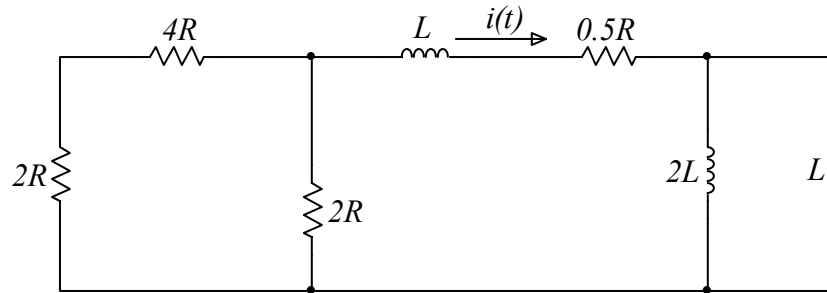After the switch is opened, the circuit becomes

Figure 13

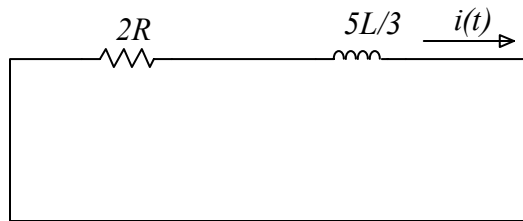By combining the resistors and the inductors the circuit reduces to

Figure 14

With the initial condition for the current $i_{t=0} = I_0 = 2\dfrac{Vs}{R}$ the solution for the current $i(t)$ becomes

$$i(t) = 2\frac{Vs}{R}e^{\frac{-6R}{5L}t} \qquad (0.21)$$

For this example we have been able to combine the inductances into an equivalent inductance and thus derive the first order differential equation for the behavior of the circuit.
However, this reduction is not possible in general. If after the reduction more than one reactive element remains in the circuit the order of the system differential equation is equal to the number of reactive elements. We will discuss the transient behavior of these higher order systems next class.

Forced Response of RC Circuits

For the circuit shown on Figure 15 the switch is closed at *t=0*. This corresponds to a step function for the source voltage Vs as shown on Figure 16.We would like to obtain the capacitor voltage *vc* as a function of time. The voltage across the capacitor at *t=0* (the initial voltage) is *Vo*.
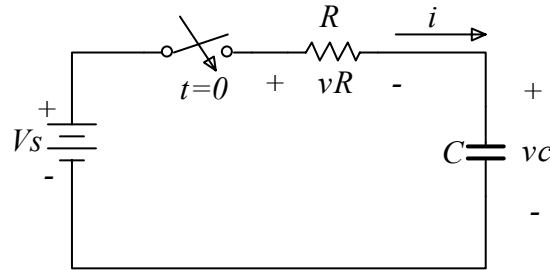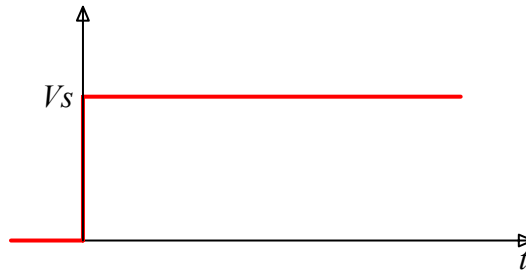

Figure 15

Figure 16

The equation that describes the system is obtained by applying KVL around the mesh.

$$vR(t) + v_c(t) = Vs \tag{0.22}$$

Which by using the current-voltage relationships becomes

$$RC \frac{dv_c(t)}{dt} + v_c(t) = Vs \tag{0.23}$$

By setting $\tau = RC$, the time constant of the circuit, Equation (0.23) becomes

$$\tau \frac{dv_c(t)}{dt} + v_c(t) = Vs \tag{0.24}$$

The solution of this equation is the combination (superposition) of the homogeneous solution $v_{ch}(t)$ and the particular solution $v_{cp}(t)$.

$$v_c = v_{ch} + v_{cp} \tag{0.25}$$

The homogeneous solution satisfies the equation

$$\tau \frac{dv_{ch}(t)}{dt} + v_{ch}(t) = 0 \tag{0.26}$$

And the particular solution the equation

$$\tau \frac{dv_{cp}(t)}{dt} + v_{cp}(t) = Vs \tag{0.27}$$

The homogeneous equation corresponds to the source free problem already investigated and its solution is

$$v_{ch}(t) = Ae^{\frac{-t}{\tau}} \tag{0.28}$$

The constant A is undefined at this point but any value will satisfy the differential equation.

The particular solution is found by inspection to be

$$v_{cp} = Vs \tag{0.29}$$

And thus the total solution becomes

$$v_c(t) = Vs + Ae^{\frac{-t}{\tau}} \tag{0.30}$$

The constant A may now be determined by considering the initial condition of the capacitor voltage. The initial capacitor voltage is Vo and thus $A=Vo-Vs$.

And the complete solution is

$$v_c(t) = Vs + (Vo - Vs)e^{-\frac{t}{\tau}} \tag{0.31}$$

Figure 17 shows the plot of *vc(t)* for *Vo=1 Volt, Vs=5 Volt* as a function of the normalized quantity t/τ. Note that after 5 time constants the voltage *vc* is within 99% of the voltage *Vs*.
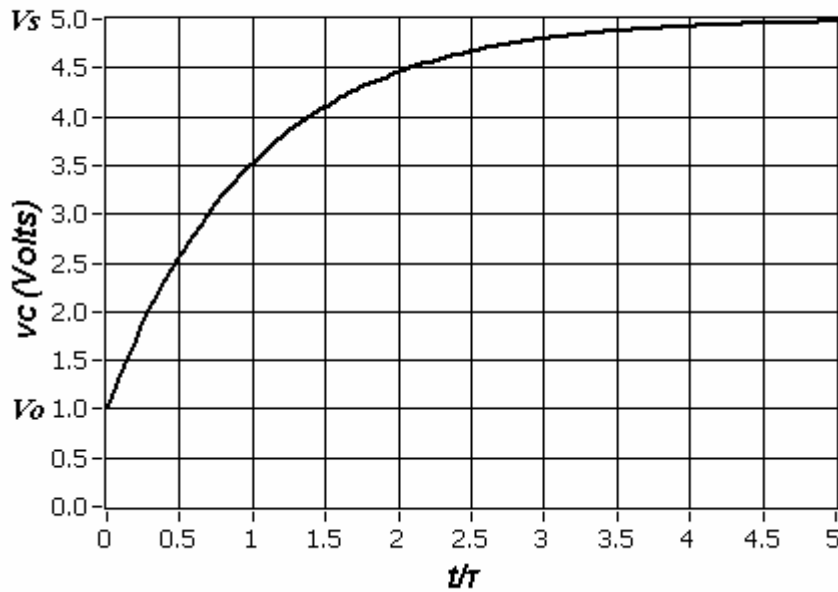
Figure 17

Now let's consider the *RC* circuit shown on Figure 18. The switch has been at position *a* for a log time and thus there is no voltage across the capacitor plates at time *t=0*. At time *t=0* the witch is moved from position *a* to position *b* where it stays for time *t1* and subsequently returned to position *a*. This switch action corresponds to the rectangular pulse shown on Figure 19.
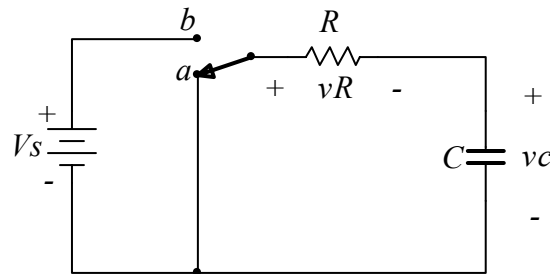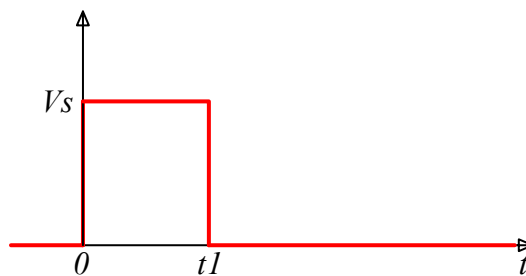

Figure 18

Figure 19

We would like to obtain the voltage *vc(t)*.

First we know that the initial condition is $vc_{t=0} = 0$. We also know that after a long time (*t>>t1*) the voltage will go back to zero. The solution of the system will tell us the evolution of the voltage *vc* from time *t=0* to *t=t1* and for *t>t1*.

The solution for *t1>t>0* is

$$v_c(t) = Vs(1 - e^{-\frac{t}{\tau}})$$ (0.32)

For *t>t1* the solution is determined by considering as the initial condition, the voltage across the capacitor at *t=t1*.

$$v_c(t1) = Vs(1 - e^{-\frac{t1}{\tau}})$$ (0.33)

And the solution for *t>t1* is

$$v_c(t) = Vs(1 - e^{-\frac{t1}{\tau}})e^{-\frac{t}{\tau}}$$ (0.34)

Figure 20 shows the complete evolution of the voltage *vc* where we have taken *t1=2τ*.
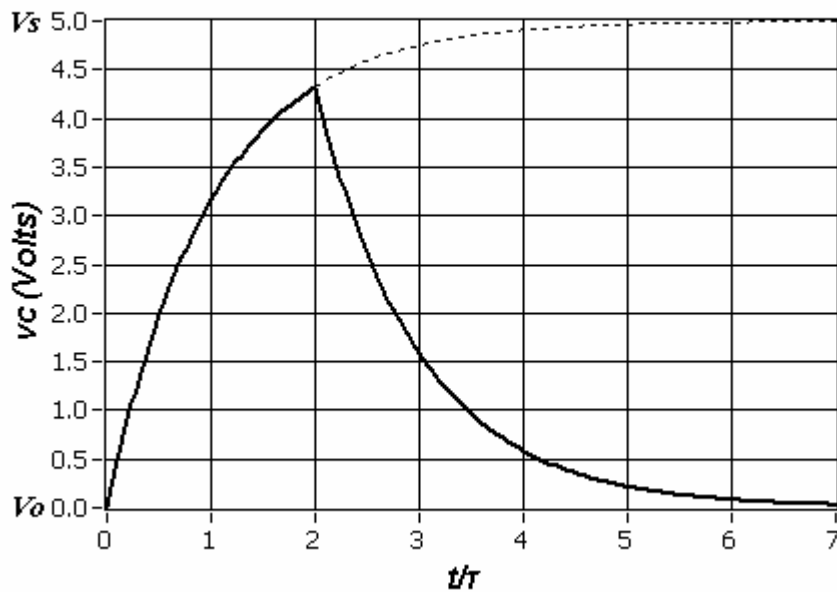

Figure 20

Procedure for transient analysis of RC and RL circuits.

1. **Determine the equivalent inductance/capacitance (L_{eq}, C_{eq})**

2. **Determine the Thevenin equivalent resistance, R_{eq}, seen by (L_{eq}, C_{eq})**

3. **The characteristic time is now known $\tau = R_{eq}C_{eq}$ or $\tau = \dfrac{L_{eq}}{R_{eq}}$**

4. **Calculate the initial value for the voltage/current flowing in the circuit**
   a. **Capacitor acts as an open circuit under dc conditions**
      i. **For a transition happening at $t = 0$, $vc(t = 0^-) = vc(t = 0^+)$**
   b. **Inductor acts as a short circuit under dc conditions**
      i. **For a transition happening at $t = 0$, $i_L(t = 0^-) = i_L(t = 0^+)$**

5. **Estimate the value of vc, i_L as $t \to \infty$ (final value)**

6. **The complete solution is:**

$$\text{solution} = \text{final value} + \left[\text{initial value - final value}\right]e^{\frac{-t}{\tau}}$$

$$vc(t) = vc_{(t \to \infty)} + \left[vc_{(t=0^+)} - vc_{(t \to \infty)}\right]e^{\frac{-t}{\tau}}$$

$$i_L(t) = i_{L(t \to \infty)} + \left[i_{L(t=0^+)} - i_{L(t \to \infty)}\right]e^{\frac{-t}{\tau}}$$

The operation of the circuit shown on Figure 21 is similar to the one discussed above. The switch has been at position *a* for a log time and thus there is no voltage across the capacitor plates at time *t=0*. At time *t=0* the witch is moved from position *a* to position *b* where it stays for time *t1* and subsequently returned to position *a*. This switch action creates the rectangular pulse shown on Figure 19.
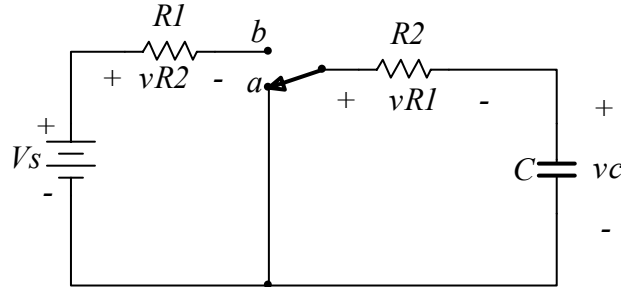

Figure 21

The *RC* circuit shown on has two time constants.
For *0<t<t1* the time constant is $\tau1 = (R1 + R2)C$.
For *t>t1* the time constant is $\tau2 = R2C$.

The solution now is

$$v_c(t) = Vs(1 - e^{-\frac{t}{\tau1}}) \qquad\qquad \text{For} \quad t \le t1$$

$$v_c(t) = Vs(1 - e^{-\frac{t1}{\tau1}})e^{-\frac{t}{\tau2}} \qquad \text{For} \quad t > t1$$

(0.35)

The plot of *vc(t)* is shown on Figure 22 for $R1 = R2$, $\tau1 = 2(\tau2)$ and $t1 = 2(\tau1)$
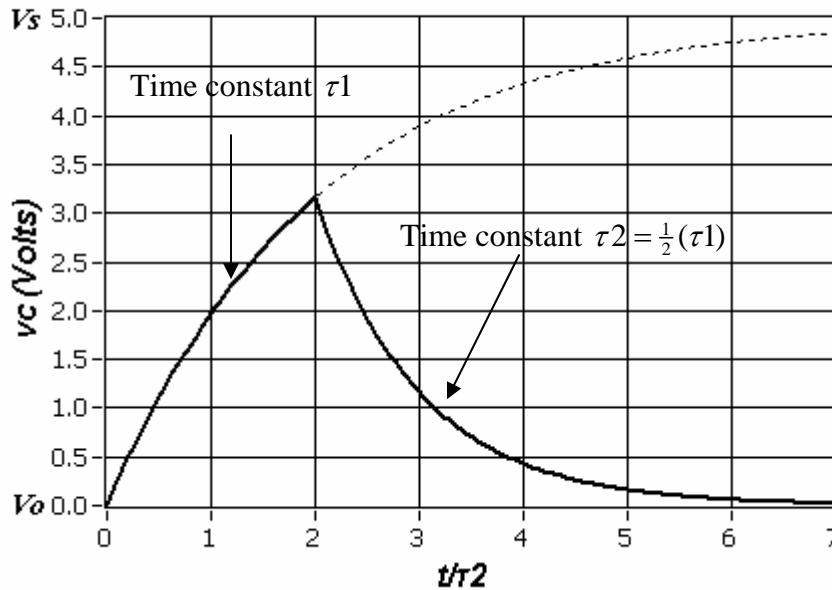

Figure 22

Problems.

The fuse element is a resistor of resistance *Rf* which is destroyed when the current through it exceeds a certain value.

The switch in the circuit has been in the closed position for a long time. At time *t=0* the switch is opened. If the maximum current that can flow through the fuse is *Im,* calculate the minimum resistance of the fuse (*Rf*) as a function of *Im* and the other circuit parameters.
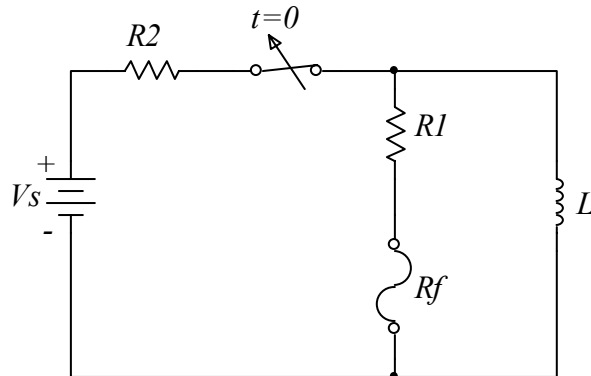
Determine R1 and R2 so that $vR_{(t=o^+)} = 2\ Volts$ and $vR_{(t=1ms)} = 1\ Volts$