MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Notes from Recitation 2

1 Problem 1: Coin Tosses

Show that the set S of sequences where the average number of heads has a limit $l \in [0, 1]$ is measurable.

Solution:

Let's rewrite the definition of limit as a set of countable constraints:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \omega_i,$$

exists and equals l if and only if for any positive integer m, there exists some N such that for all $n \geq N$, we have

$$\Big|\frac{1}{n}\sum_{i=1}^{n} \omega_i - l\Big| < \frac{1}{m}.$$

It's not hard to see that this is equivalent to the standard definition of the limit. Let's define the sets

$$S_{\varepsilon,N} = \left\{\omega \,:\, \Big| \text{ for all } n \geq N,\ |\frac{1}{n}\sum_{i=1}^{n} \omega_i - l\Big| < \varepsilon\right\}.$$

Rewriting the above limit definition by replacing "exists" and "for all" operators with their equivalent unions and intersections, we get

$$S = \cap_{m=1}^{\infty} \cup_{N=1}^{\infty} S_{1/m,N}.$$

Now define sets:

$$S_{\varepsilon,n} \quad = \quad \left\{\omega \,:\, \Big|\frac{1}{n}\sum_{i=1}^{n} \omega_i - l\Big| < \varepsilon\right\}.$$

Then it is clear that $S_{\varepsilon,n} \in \mathcal{F}_n$. Furthermore,

$$S_{1/m,N} = \bigcap_{n=N}^{\infty} S_{1/m,n}.$$

so putting it all together, we have

$$S = \bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} S_{\frac{1}{m},n},$$

and thus $S \in \mathcal{F}$.

2 Problem 2: Coin Tosses

Let $\mathcal{F}$ be the σ-field corresponding to the infinite coin toss model. Let S be the set of sequences such that there exists some time t, so that the number of heads that come after time t is always greater than or equal to the number of tails that come after time t. (So it does not matter what happens before time t). Show that $S \in \mathcal{F}$.

Solution: In symbols, we have

$$S = \left\{ \omega \,:\, \exists t \,\text{s.t.} \sum_{j=t+1}^{t+k} w_j \geq \lceil \frac{k}{2} \rceil, \quad \forall k \right\}.$$

Also define the sets:

$$\begin{aligned} S_t &= \left\{ \omega \,:\, \sum_{j=t+1}^{t+k} w_j \geq \lceil \frac{k}{2} \rceil, \quad \forall k \right\} \\ S_{t,k} &= \left\{ \omega \,:\, \sum_{j=t+1}^{t+k} w_j \geq \lceil \frac{k}{2} \rceil \right\}. \end{aligned}$$

Now observe that $S_{t,k} \in \mathcal{F}_{t+k}$, and hence $S_{t,k} \in \mathcal{F}$. In addition, we have

$$S_t = \bigcap_{k=1}^{\infty} S_{t,k},$$

and also

$$\begin{aligned} S &= \bigcup_{t=1}^{\infty} S_t \\ &= \bigcup_{t=1}^{\infty} \bigcap_{k=1}^{\infty} S_{t,k}, \end{aligned}$$

and therefore $S \in \mathcal{F}$. Note how the infinite intersections and unions correspond to the operators "$\forall$" and "$\exists$."

3 Borel-Cantelli

Here are a few simple observations related to the Borel-Cantelli lemma.

1. Consider an infinite sequences of independent coin tosses, with probability of landing on heads being $p > 0$ (this model was constructed rigorously in lecture 2). Can we show that a head occurs infinitely often with probability 1?

This follows from the Borel-Cantelli lemma since $\sum_{i=1}^{\infty} p = \infty$, and distinct coin tosses are independent.

2. Can the same be said for any pattern, that is, any pattern occurs infinitely often with probability 1? A pattern is understood to mean something like 5 heads, followed by 17 tails, followed by 3 heads.

This can be done by the same argument as above. Let K be some number so that the pattern has a chance $p^* > 0$ of occuring in the first K flips. In the above example, we can pick $K = 25$, and the probability of the pattern occuring is $p^* = p^5(1-p)^{17}p^3$. Let A_i be the event that the pattern occurs in the i'th block of K flips. The Borel-Cantelli lemma gives again that $\sum_{i=1}^{\infty} p^* = \infty$, and since A_i are independent, we can conclude they occur infinitely often.

3. Consider two scenarios. In the first, the probability of landing on heads in the i'th toss is $1/n$. In the second, its $1/n^2$. As before, coin tosses are independent.

A fact from analysis is that while

$$\sum_{i=1}^{n} \frac{1}{n} = \infty,$$

its true that

$$\sum_{i=1}^{n} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This implies that in the first scenario, the coin lands on heads infinitely often, while in the second scenario, heads only occurs finitely often.