MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.097 (UG) Fundamentals of Photonics
6.974 (G) Quantum Electronics

Spring 2006

Quiz II

Time: April 21*, 2006, 2-3:30pm*

Problems marked with (Grad) are for graduate students only.

- This is a closed book exam, but two 81/2"x11" sheets (both sides) is allowed.
- At the end of the booklet there is a collection of equations you might find helpful for the exam.
- Everything on the notes must be in your original handwriting (i.e. material cannot be Xeroxed).
- You have 90 minutes for this exam.
- There are 5 problems for undergraduate and 6 problems for graduate students on the exam with the number of points for each part and the total points for each problem as indicated. Note, that the problems do not all have the same total number of points.
- Some of the problems have parts for graduate students only. Undergraduate students solving these problems can make these additional points and compensate eventually for points lost on other problems.
- Make sure that you have seen all 22 numbered sides of this answer booklet.
- The problems are not in order of difficulty. We recommend that you read through all the problems, then do the problems in whatever order suits you best.
- We tried to provide ample space for you to write in. However, the space provided is not an indication of the length of the explanation required. Short, to the point, explanations are preferred to long ones that show no understanding.
- Please be neat-we cannot grade what we cannot decipher.

All work and answers must be in the space provided on the exam booklet. You are welcome to use scratch pages that we provide but when you hand in the exam we will not accept any pages other than the exam booklet.

Exam Grading

In grading of the exams we will be focusing on your level of understanding of the material associated with each problem. When we grade each part of a problem we will do our best to assess, from your work, your level of understanding. On each part of an exam question we will also indicate the percentage of the total exam grade represented by that part, and your numerical score on the exam will then be calculated accordingly.

Our assessment of your level of understanding will be based upon what is given in your solution. A correct answer with no explanation will not receive full credit, and may not receive much-if any. An incorrect final answer having a solution and explanation that shows excellent understanding quite likely will receive full (or close to full) credit.

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No work on this page will be evaluated.

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Quiz II

Time: April *21, 2006, 2-3:30pm*

Full Name: ____

Are you taking 6.974 ____ or 6.097 ____ ?

Problem	Max points (undergrad.)	Max points (grad.)	Your Points
1. Gaussian Beams and Resonators	21	36	
2. Coupled Waveguide Modes	20	20	
3. Polarization of Light	14	14	
4. Thermal Radiation	18	18	
5. Schrödinger's Equation	17	17	
6. Superposition State	-	15	
Total	**90**	**120**	

Problem 1: Gaussian Beams and Resonators

(21 points undergrad/36 points grad)

(a) (4 points) What are the characteristics of the Gaussian beam at the waist?

(b) (3 points) What is the physical meaning of the Rayleigh range z_R of a Gaussian beam?

(c) (5 points) In the resonator shown below the spot size of the mode is larger at mirror 2. Is the waist of the beam situated closer to mirror 1 or mirror 2? Why?

(d) (9 points) Consider the stable plano-concave resonator shown below. The fundamental mode of the resonator at wavelength $\lambda = 1\mu m$ is a Gaussian beam with the waist radius w_0.

– How will the waist radius of the mode change if λ is increased to $10\mu m$? Show your reasoning.

– How will the divergence angle of the beam change if λ is increased to $10\mu m$?

(e) **<u>(Grad)</u>** (15 points) Suppose you need to design a *two-mirror* resonator for a high-power laser operating at $\lambda = 1\mu m$. You have at your disposal 3 pairs of mirrors: *two* spherical mirrors with *ROC* (radius of curvature) of $1m$, *two* spherical mirrors with $ROC = 2m$, and *two* plane mirrors ($ROC = \infty$). There are two requirements that must be satisfied:

(i) for a given power, the intensity at the mirrors must be as small as possible to avoid mirror damage;

(ii) to achieve most effective amplification, the gain medium with radius $r = 0.7mm$ is placed into the *waist region* of the Gaussian beam. The waist size must not be too small, otherwise the beam will fill just a part of the gain medium, and it also must not be too large, otherwise only a fraction of the beam will be amplified. Suppose that the optimum waist radius is half the gain medium radius, $w_0 = r/2 = 0.35mm$. Thus the waist radius of the beam is fixed by the gain medium and cannot be varied in resonator design.

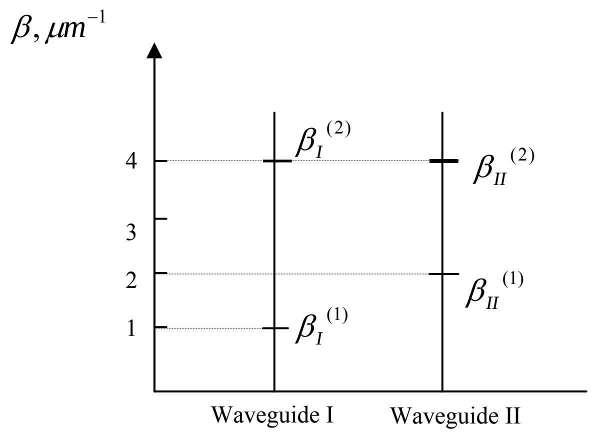
– Which two of the available mirrors would you choose for the resonator? Why?

– Approximately what length of resonator would you choose?

Problem 2: Coupled Waveguide Modes

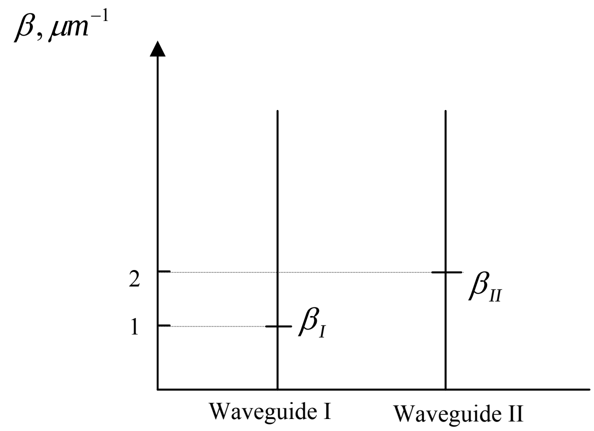
(20 points)

Two waveguides are spaced closely to each other. In the absence of the other waveguide, the unperturbed waveguides each have two guided modes with the propagation constants sketched below (note that the subscripts denote the waveguide number and superscripts denote the mode number).



a) (5 points) Which pair of modes can exchange full power upon arbitrarily small coupling between these waveguides? Give a brief explanation.

b) (10 points) Now consider that in the absence of the other waveguide, the unperturbed waveguides each have only one guided mode with the propagation constants sketched below. Power is launched in waveguide I with wave number β_I and there is coupling to the guided mode in waiveguide II with the wave number β_{II}. The coupling is described by the coupling coefficient $\kappa_{I-II} = 0.5\mu m^{-1}$. What will be the maximum power coupling from waveguide I to waveguide II?



c) (5 points) At what length L of the two waveguides does the maximum power transfer from waveguide I to waveguide II occur according to part (b)?

Problem 3: Polarization of Light

(14 points)

A plane electromagnetic wave propagates in free space along the positive z-axis. The electric field vector of the wave is given as

$$\vec{E}(z,t) = E_{0x}\cos(\omega t - kz)\vec{e}_x + E_{0y}\cos(\omega t - kz + \varphi)\vec{e}_y \tag{1}$$

(a) (4 points) What is the polarization in the following cases, i.e. is the light linearly polarized, circularly polarized, or elliptically polarized?

1) $\varphi = 0$, $E_{0x} \neq 0$, and $E_{0y} = 0$;

2) $\varphi = \pi/2$ and $E_{0x} = E_{0y}$.

(b) (2 points) Determine a Jones vector for each of the two cases in part (a).

(c) (2 points) Suppose we have a half-wave plate with a phase retardation of $\Gamma = \pi$ and the principal axes rotated by an angle $\psi = 45°$ with respect to the x- and y- axes. What is the Jones Matrix of this half-wave plate?

(d) (6 points) What is the final polarization of the two cases in part (a) after propagation through the half-wave plate of part (c)?

Problem 4: Thermal Radiation

(18 points)

We are given a system in an Ulbricht sphere of atoms and photons in thermal equilibrium at a temperature T=300K. We will examine the photons that are emitted and absorbed at a wavelength of 620nm.

(a) (6 points) What kind of radiator is described above? Why?

(b) (4 points) Is 620nm the part of the energy spectrum where the atoms emit the most electromagnetic energy?

(c) (4 points) What is the average number of photons you expect to be stored in a radiation mode of this system at 620nm?

(d) (4 points) Now assume that our system only had two energy levels. Show that at high temperatures the population densities n_1 and n_2 of the two states tend to become equal.

Problem 5: Schrödinger's Equation

(17 points)

A particle of mass m that moves in a potential $V(x)$ is in the state

$$\Psi(x,t) = \sqrt[4]{\frac{2a}{\pi}} e^{-ax^2 - jbt} ,$$

where a and b are positive real constants.

(a) (5 points) Find the uncertainty in particle position.

(b) (7 points) Is this a stationary state? Why or why not? If this is a stationary state, find the energy of the system.

(c) (5 points) Find the probability current, $J(x, t)$.

Problem 6: Superposition State

(<u>Graduate problem</u>, 15 points)

Consider a particle in an infinite 1-D square well with potential

$$V(x) = \begin{cases} 0, & |x| \le a/2 \\ V_0 = \infty, & |x| > a/2 \end{cases}$$

The particle is initially in a superposition state of the first two stationary states, i.e.

$$\Psi(x, t=0) = \frac{1}{\sqrt{2}}\left[\psi_1(x) + \psi_2(x)\right],$$

where the wave functions are $\psi_1(x) = \sqrt{\frac{2}{a}}\cos\left(\frac{\pi}{a}x\right)$ and $\psi_2(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{2\pi}{a}x\right)$ inside the well and zero outside the well. For a given state, n, the corresponding energy eigenvalue is $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$.

What is the expected value of the particle position, x, for $t > 0$?

Note that the equation sheet at the back of this booklet contains some formulas you may find useful for this problem.

Quiz I Equation Sheet

Maxwell's Equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$

$$\nabla \cdot \vec{D} = \rho \qquad \nabla \cdot \vec{B} = 0$$

Material Equations

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon \vec{E} \qquad \vec{P} = \varepsilon_0 \chi_e \vec{E} \qquad \varepsilon = \varepsilon_0 (1 + \chi_e)$$

$$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M} = \mu \vec{H} \qquad \vec{M} = \chi_m \vec{H} \qquad \mu = \mu_0 (1 + \chi_m)$$

Index of Refraction $\quad n^2 = 1 + \chi$, $\quad$ for $\chi << 1$: $\quad n \approx 1 + \chi / 2$

Poynting Vector

$$\vec{S} = \vec{E} \times \vec{H} \qquad \vec{T} = \frac{1}{2} \underline{\vec{E}} \times \underline{\vec{H}}^*$$

Energy density

$$w_e = \frac{1}{2} \varepsilon \vec{E}^2 \qquad w_m = \frac{1}{2} \mu \vec{H}^2 \qquad w = w_e + w_m$$

Snell's Law

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

Brewster's Angle

$$\tan \theta_B = \frac{n_2}{n_1}$$

Reflectivity

$$r^{TE} = \frac{Z_2^{TE} - Z_1^{TE}}{Z_1^{TE} + Z_2^{TE}} \qquad r^{TM} = \frac{Z_1^{TM} - Z_2^{TM}}{Z_1^{TM} + Z_2^{TM}}$$

$$Z_{1/2}^{TE} = \sqrt{\frac{\mu_{1/2}}{\varepsilon_{1/2}}} \frac{1}{\cos \theta_{1/2}} \qquad Z_{1/2}^{TM} = \sqrt{\frac{\mu_{1/2}}{\varepsilon_{1/2}}} \cos \theta_{1/2}$$

$$r^{TE} = \frac{n_1 \cos \theta_1 - n_2 \cos \theta_2}{n_1 \cos \theta_1 + n_2 \cos \theta_2} \qquad r^{TM} = \frac{\dfrac{n_2}{\cos \theta_2} - \dfrac{n_1}{\cos \theta_1}}{\dfrac{n_2}{\cos \theta_2} + \dfrac{n_1}{\cos \theta_1}}$$

Transmitivity

$$t^{TE} = \frac{2 Z_2^{TE}}{Z_1^{TE} + Z_2^{TE}} \qquad t^{TM} = \frac{2 Z_1^{TM}}{Z_1^{TM} + Z_2^{TM}}$$

$$t^{TE} = \frac{2 n_1 \cos(\theta_1)}{n_1 \cos(\theta_1) + n_2 \cos(\theta_2)} \qquad t^{TM} = \frac{2 \dfrac{n_2}{\cos(\theta_2)}}{\dfrac{n_2}{\cos(\theta_2)} + \dfrac{n_1}{\cos(\theta_1)}}$$

Power Refl. Coef.

$$R^{TE} = \left| r^{TE} \right|^2 \qquad R^{TM} = \left| r^{TM} \right|^2$$

Power Transm. Coef.

$$T^{TE} = \left| t^{TE} \right|^2 \frac{Z_1^{TE}}{Z_2^{TE}} = \frac{4 Z_1^{TE} Z_2^{TE}}{\left| Z_1^{TE} + Z_2^{TE} \right|^2} \qquad T^{TM} = \left| t^{TM} \right|^2 \frac{Z_2^{TM}}{Z_1^{TM}} = \frac{4 Z_1^{TM} Z_2^{TM}}{\left| Z_1^{TM} + Z_2^{TM} \right|^2}$$

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Pulse Dispersion $$\frac{\partial A(z,t')}{\partial z} = j\frac{k''}{2}\frac{\partial^2 A(z,t')}{\partial t'^2}$$

Gaussian Pulse $$\tau(L) = \tau\sqrt{1+\left(\frac{k''L}{\tau^2}\right)^2} \qquad \tau_{FWHM} = 2\sqrt{\ln 2}\,\tau$$

Fabry Perot $$\left|S_{21}\right|^2 = \frac{(1-R)^2}{(1-R)^2 + 4R\sin^2(\phi/2)} \qquad \text{where } \phi = 2kL\,,\ k = \frac{2\pi f}{c_0}n$$

Beam Splitter S-matrix $S = \begin{pmatrix} r & jt \\ jt & r \end{pmatrix}$, with $r^2 + t^2 = 1$

Constants

$\varepsilon_o = 8.85 \times 10^{-12}$ C^2/Nm^2	permittivity of free space
$\mu_o = 4\pi \times 10^{-7}$ N/A^2	permeability of free space
$m = 9.11 \times 10^{-31}$ kg	mass of an electron
$e = 1.60 \times 10^{-19}$ C	charge of an electron

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Quiz II   Equation Sheet

Gaussian Beams

$$E(r,z) = \frac{\sqrt{2P}}{\sqrt{\pi}\, w(z)} \exp\left[-\frac{r^2}{w^2(z)} - jk_0 \frac{r^2}{2R(z)} + j\zeta(z) \right]$$

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R} \right)^2} \qquad R(z) = z \left[1 + \left(\frac{z_R}{z} \right)^2 \right]$$

$$\zeta(z) = \arctan\left(\frac{z}{z_R} \right)$$

$$I(r,z) = I_0 \frac{w_0^2}{w(z)^2} \exp\left[-\frac{2r^2}{w^2(z)} \right]$$

$$z_R = \frac{k w_0^2}{2} = \frac{\pi w_0^2}{\lambda} \qquad \theta = \frac{\lambda}{\pi w_0}$$

q-parameter

$$q(z) = z + jz_R \qquad \frac{1}{q(z)} = \frac{1}{R(z)} - j\frac{\lambda}{\pi w^2(z)}$$

$$E(r,z) = \frac{1}{q(z)} \exp\left[-jk_0 \frac{r^2}{2q(z)} \right]$$

Resonator Stability

$$0 \le g_1 g_2 \le 1 \qquad g_1 = 1 - \frac{L}{R_1} \qquad g_2 = 1 - \frac{L}{R_2}$$

Waveguide Coupling

$$P_1(z) = P_1(0) \left(\cos^2 \gamma z + \left(\frac{\Delta\beta}{\gamma} \right)^2 \sin^2 \gamma z \right)$$

$$P_2(z) = P_1(0) \frac{|\kappa_{21}|^2}{\gamma^2} \sin^2(\gamma z)$$

$$\gamma = \sqrt{\Delta\beta^2 + |\kappa_{12}|^2} \qquad \Delta\beta = \frac{\beta_1 - \beta_2}{2} \qquad \beta_0 = \frac{\beta_1 + \beta_2}{2}$$

Polarization, Retardation Plate

$$W = \begin{pmatrix} e^{-j\Gamma/2}\cos^2\psi + e^{j\Gamma/2}\sin^2\psi & -j\sin\left(\frac{\Gamma}{2}\right)\sin(2\psi) \\ -j\sin\left(\frac{\Gamma}{2}\right)\sin(2\psi) & e^{-j\Gamma/2}\sin^2\psi + e^{j\Gamma/2}\cos^2\psi \end{pmatrix}$$

(ψ is the rotation angle of the principal axes with respect to the x- and y- axes)

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if $\Gamma = \pi$: $$W = -j\begin{pmatrix} \cos(2\psi) & \sin(2\psi) \\ \sin(2\psi) & -\cos(2\psi) \end{pmatrix}$$

if $\Gamma = \frac{\pi}{2}$: $$W = \begin{pmatrix} \frac{1}{\sqrt{2}}[1 - j\cos(2\psi)] & -j\frac{1}{\sqrt{2}}\sin(2\psi) \\ -j\frac{1}{\sqrt{2}}\sin(2\psi) & \frac{1}{\sqrt{2}}[1 + j\cos(2\psi)] \end{pmatrix}$$

Wien's Law $$w(f) = \frac{8\pi h f^3}{c^3} e^{-hf/kT}$$

Wien Displacement Law $$\lambda_{\max} = \frac{hc}{4.965kT}$$

Rayleigh-Jeans Law $$w(f) = \frac{8\pi}{c^3} f^2 kT$$

Planck's Law $$w(f) = \frac{8\pi f^2}{c^3} \frac{hf}{\exp\frac{hf}{kT} - 1}$$

Einstein's A and B Coefficients $$A_{21} = \frac{8\pi f_{21}^3}{c^3} B_{12} \qquad B_{21} = B_{12}$$

Schrödinger's Equation $$j\hbar\frac{d\Psi(r,t)}{dt} = -\frac{\hbar^2}{2m}\Delta\Psi(r,t) + V(r)\Psi(r,t)$$

Heisenberg's Uncertainty Principle $$\Delta x \Delta p \geq \frac{\hbar}{2}$$

de Broglie's Formula $$p = \hbar k$$

Einstein's Energy/Frequency Relation $$E = \hbar\omega$$

Trigonometric Identities $$\sin 2\alpha = 2\sin\alpha\cos\alpha \qquad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\sin\alpha\sin\beta = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)$$

$$\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta)$$

$$\sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta)$$

$$\cos\alpha\sin\beta = \frac{1}{2}\sin(\alpha+\beta) - \frac{1}{2}\sin(\alpha-\beta)$$

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Helpful Integrals

$$\int_{-\infty}^{\infty} e^{-a x^2}\,dx = \sqrt{\frac{\pi}{a}} \qquad \int_{-\infty}^{\infty} x e^{-a x^2}\,dx = 0$$

$$\int_{-\infty}^{\infty} x^2 e^{-a x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

$$\int x\sin(\alpha x)dx = \frac{\sin(\alpha x) - \alpha x\cos(\alpha x)}{\alpha^2} + C$$

$$\int x\cos(\alpha x)d\alpha = \frac{\cos(\alpha x) + \alpha x\sin(\alpha x)}{\alpha^2} + C$$

Constants

$k = 1.380650 \times 10^{-23}$ J/K	Boltzmann's constant
$c_0 = 2.997925 \times 10^{8}$ m/s	Speed of light in free space
$h = 6.626068 \times 10^{-34}$ J·s	Planck's constant
$\hbar = 1.054571 \times 10^{-34}$ J·s	