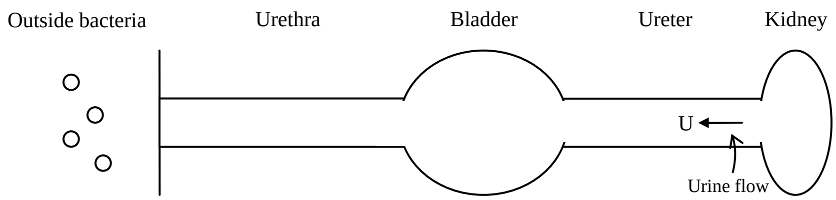
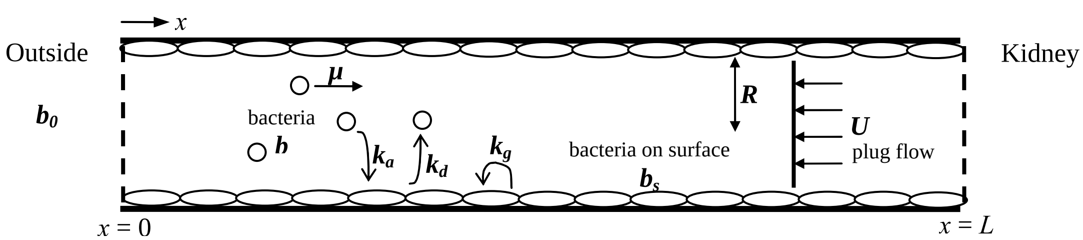
Final Review Session Example Problem
Bacterial Migration during Urinary Tract Infection

12/03/2004



Urinary tract infection (UTI) is a family of inflammatory infections of the urethra, bladder, ureter, or the kidney, caused by exogenous bacteria that migrate through the urinary tract (see above). One of the most severe types of UTI is pyelonephritis – inflammatory infection of the kidney – whose infection process we would like to understand with our modeling efforts. In our approach, we assume that the only defense mechanism is through the flow of urine, which delays the bacterial migration towards the kidney. Further, we will ignore infections of the urethra, bladder, and ureter and regard the urinary tract as a long, cylindrical tube:



Let b be the bacteria in solution/fluid in units of [moles/volume] and b_s be the bacteria on the tract wall in units of [moles/area]. Bacteria in fluid b migrate with migration coefficient μ. They also tend to attach to the wall and dissociate from the wall with reaction rates k_a and k_d, which take into account the impediment of wall attachment and facilitation of detachment through the flow of urine:

$$k_a = \frac{k_{a,0}}{U_0 + U} \qquad\qquad k_d = k_{d,0}\left(U_0 + U\right)$$

where U_0 is some basal fluid flow rate, $k_{a,0}$ and $k_{d,0}$ are reaction constants in absence of fluid flow. Once attached to the tract wall, bacteria can grow with a growth rate of k_g. Let's assume that the bacterial migration on the wall is negligible compared to the migration in the fluid at all times. For simplicity, the urine flow is modeled as plug flow.

Our overall goal is to find the bacterial flux into the kidney.

SUMMARY OF ANSWER FORMULATION

Assumptions: ***R* << *L*** (1-dimensional problem)

Reactions:

$$R_b = \frac{\partial b}{\partial t} = k_d b_s - k_a b$$

$$R_{b_s} = \frac{\partial b_s}{\partial t} = -k_d b_s + k_a b + k_g b_s$$

Governing equations:

$$\vec{N}_i = -D_i \frac{\partial c_i}{\partial x} + \vec{v} c_i$$

$$\frac{\partial c_i}{\partial t} = -\frac{\partial N_i}{\partial x} + R_v$$

$$\Rightarrow \frac{\partial b}{\partial t} = \mu \frac{\partial^2 b}{\partial x^2} + U \frac{\partial b}{\partial x} + k_d b_s - k_a b$$

Assume quasi-steady-state approximation for b_s:

$$\frac{\partial b_s}{\partial t} = -k_d b_s + k_a b + k_g b_s \approx 0 \Rightarrow b_s = \frac{k_a}{k_d - k_g} b$$

Boundary conditions:

I.C.	$b(x, t=0) = 0$		
B.C. #1	$b(x=0, t) = b_0$		
B.C. #2	$b(x=L, t) = 0$	or	$\left.\frac{\partial b}{\partial x}\right\vert_{x=L} = 0$
	This makes more sense!		w/o flux, no infection!

Normalizing equations:

$$c = \frac{b}{b_0} \qquad \xi = \frac{x}{L} \qquad \tau = \frac{U}{L^2} t$$

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial \xi^2} + Pe \frac{\partial c}{\partial \xi} + \alpha c, \text{ where } Pe = \frac{UL}{\mu} \text{ and } \alpha = \frac{L^2}{\mu}\left(\frac{k_g k_a}{k_d - k_g}\right)$$

With $c(\xi, 0) = 0$, $c(0, \tau) = 1$, $c(1, \tau) = 0$

Pe/α tells which term is dominating your problem.

Steady-state solution

$$c(\zeta) = \frac{e^{\lambda_1} e^{\lambda_2 \xi} - e^{\lambda_2} e^{\lambda_1 \xi}}{e^{\lambda_1} - e^{\lambda_2}} \text{ with } \lambda_{1,2} = -\frac{Pe}{2} \pm \sqrt{\left(\frac{Pe}{2}\right)^2 - \alpha}$$

$$N_b\Big|_{\xi=1} = -\frac{\mu b_0}{L} \frac{\partial c}{\partial \xi}\Big|_{\xi=1} - U b_0 c(\xi = 1)$$