BE.430 Tutorial:
Green's Functions

9/29/2004

General Theory

$$\mathcal{L}\left[v(\xi)\right] = f(\xi)$$

where $\mathcal{L}\left[v(\xi)\right] = a_0(\xi)\dfrac{d^2v(\xi)}{dx^2} + a_1(\xi)\dfrac{dv(\xi)}{dx} + a_2(\xi)v(\xi)$

Properties of Equation:

- Inhomogeneous boundary value problems
- Ordinary differential equations
- Nonlinear $f(\xi)$

Proposed Method of Solving:

- Find the Green's Function, $G(\xi)$, such that

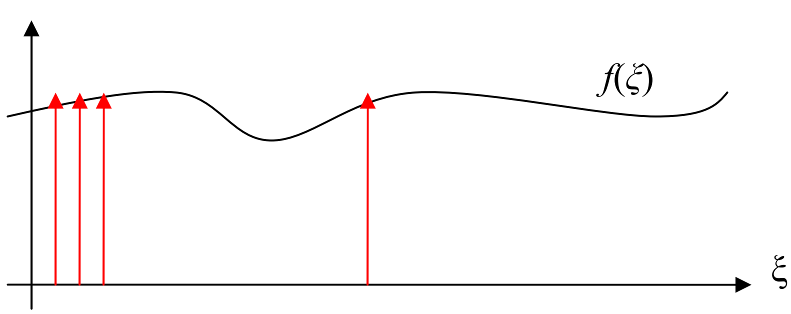
$$\mathcal{L}\left[G(\xi)\right] = \delta(\xi) \tag{1}$$

- Then, integrate $G(\xi)$ with $f(\xi)$ over entire domain of ξ to find $v(\xi)$

$$v(\xi) = \int_{\xi_1}^{\xi_2} G(\xi, z) f(z)\, dz \tag{2}$$

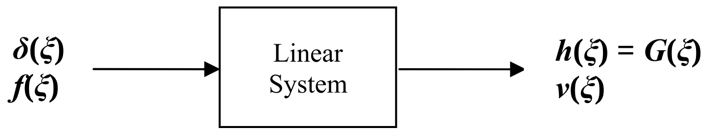
Concept of Green's Function:

Any continuous function, $f(x)$, can be represented as infinite sum of delta functions with the limit of their separation approaching zero.



$$f(\xi) = \int_{-\infty}^{\infty} \delta(\xi - z) f(z) dz \tag{3}$$

When a delta function, $\delta(\xi)$ applied to a linear system, it will generate an impulse response, $h(\xi)$. Likewise, the linear operator can be modeled as a linear system such that when it operates on the Green's Function, it will result in a delta function.



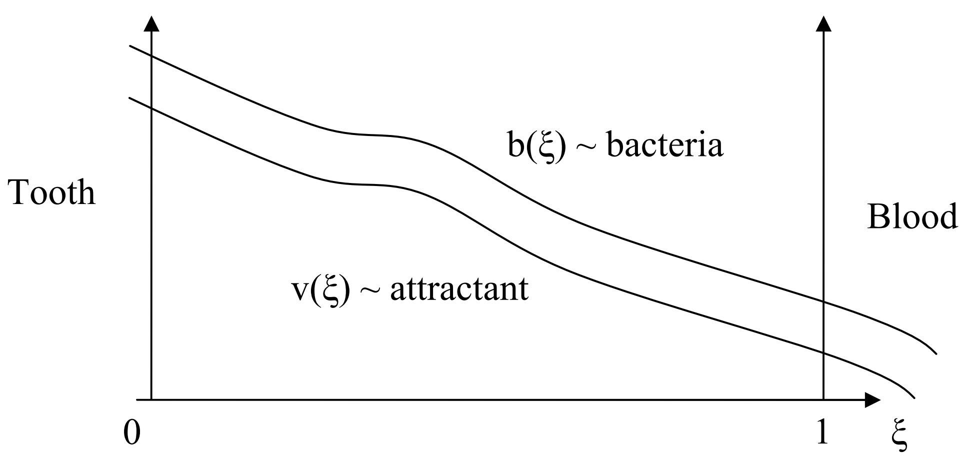
In other words, the Green function is the impulse response which describes how the system will react to a single point source. Then, the summation of all the effects that a distribution of point sources produced on the system, $f(\xi)$, will lead to the total response or output of the system.

$$v(\xi) = \int_{-\infty}^{\infty} h(\xi - z) f(z) dz \tag{4}$$

Please note the similarity in form between Eq. (2) and (4). Using infinity as limits of integration is a generalization for any impulse response, while Green's function requires that the domain be bounded.

Example discussed in class

Consider the periodontal infection model discussed in class. We are looking for the attractant profile for a given bacteria density profile.



Conservation of Species for Attractants:

$$\frac{d^2 v}{d\xi^2} - \alpha^2 v = -u(\xi) \quad \text{where} \quad u(\xi) = \frac{b(\xi)}{b^*} \text{ and } \xi = \frac{x}{L}$$

Linear Operator Parameters:

$$\mathcal{L} = \frac{d^2}{d\xi^2} - \alpha^2 \quad \text{where } a_0 = 1,\ a_1 = 0, \text{ and } a_2 = -\alpha^2$$

Boundary Conditions:

1. $\xi = 0$ $\qquad \frac{dv}{d\xi} = 0$ $\qquad$ (no flux through tooth)

For the second boundary condition, let's consider the diffusion limited case, such that

$$\frac{h_a L}{D_a} >> 1 \text{, therefore } \frac{\eta}{\Delta_a} \to \infty$$

(see class notes). Then,

2. $\xi = 1$ $\qquad v = 0$

Solve two linearly independent equations that satisfy $\mathcal{L}v = 0$:

$$v_1 = A_1 e^{\alpha\xi} + A_2 e^{-\alpha\xi} = A_1\text{'}\sinh\alpha\xi + A_2\text{'}\cosh\alpha\xi$$
$$v_2 = B_1 e^{\alpha\xi} + B_2 e^{-\alpha\xi} = B_1\text{'}\sinh\alpha\xi + B_2\text{'}\cosh\alpha\xi$$

Applying the boundary conditions:

Since there are four unknown constants and only two boundary conditions, there will be infinitely many solutions. We just need to found a set of four non-trivial constants such that v_1 satisfies boundary condition 1 and v_2 satisfies boundary condition 2. Let A_1' = 0, A_2' = 1, B_1' = -cosh(α), B_2' = 1:

$$v_1 = \cosh(\alpha\xi) \text{ and } v_2 = \sinh(\alpha(1-\xi))$$

Solution of the Green's function

$$G(\xi,z) = \begin{cases} \dfrac{v_2(z)v_1(\xi)}{a_0(z)w[v_1(z),v_2(z)]}, \xi \le z \\ \dfrac{v_1(z)v_2(\xi)}{a_0(z)w[v_1(z),v_2(z)]}, \xi > z \end{cases}$$

where $w[v_1(z), v_2(z)]$ is the Wronskian of the functions v_1 and v_2.

$$a_0(z)w[v_1(z),v_2(z)] = a_0(z)\left[v_1(z)\frac{dv_2}{d\xi}\bigg|_z - v_2(z)\frac{dv_1}{d\xi}\bigg|_z \right] = \alpha\cosh\alpha$$

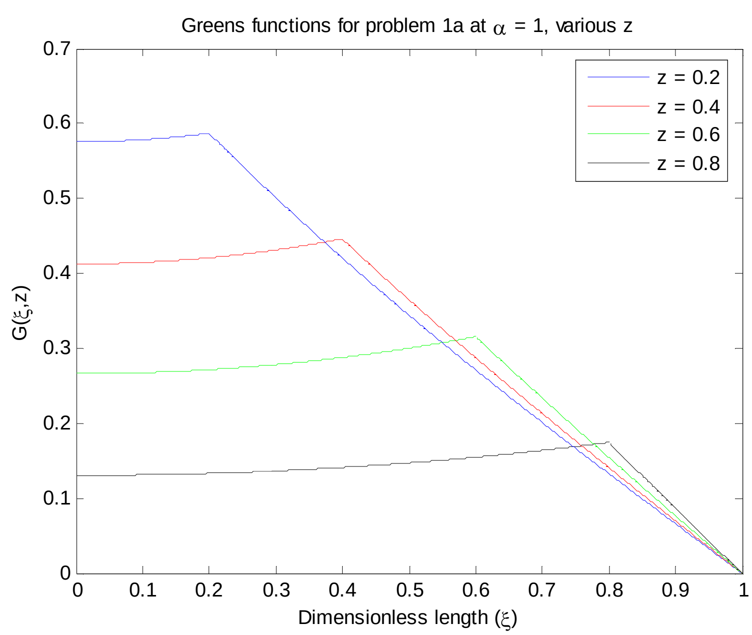
Since the denominator of the Green's Function is a constant, it means that the two solutions are self-adjoint.

Final Green's Function Expression:

$$G(\xi,z) = \begin{cases} \dfrac{\sinh\left[\alpha\left(z-1\right)\right]\cosh\left[\alpha\xi\right]}{\alpha\cosh\alpha}, \xi \le z \\ \dfrac{\cosh\left[\alpha z\right]\sinh\left[\alpha\left(\xi-1\right)\right]}{\alpha\cosh\alpha}, \xi > z \end{cases}$$

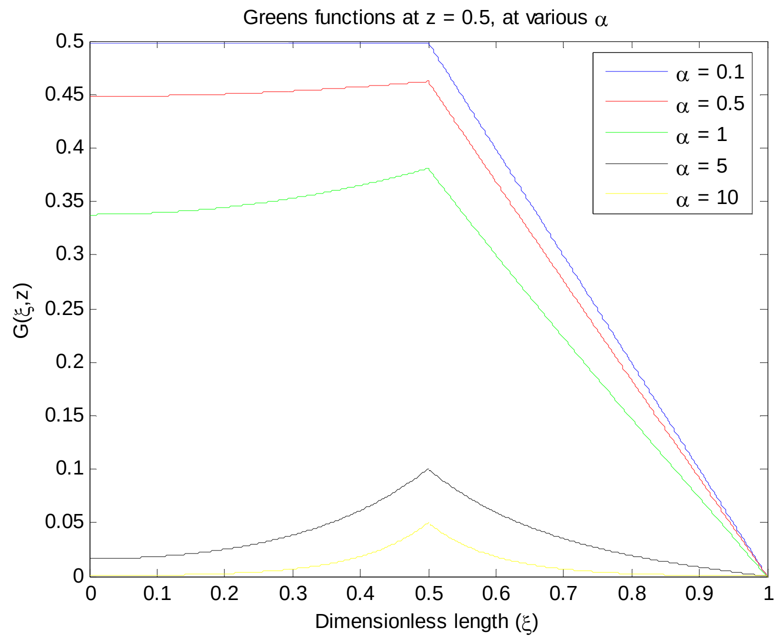
Physical meaning of the Green's Function

A linear operator defines the physics of the space. Our linear operator indicates that there is diffusion and first-order reaction throughout the system in which the linear operator is defined. The Green's Function describes the profile of v if only a point source of bacteria generating attractants is present at some z. The sum of the sources across all z describes the actual value of the attractant at a point ξ.



The plot above shows that at a given point source, there is a balance between diffusion and first-order consumption satisfying both boundary condition (zero gradient left, $G = 0$ right). A species right of the source will decrease due to diffusion and consumption to satisfy zero-concentration at the right boundary. A species left of the source will decrease so that the no flux condition is satisfied at the left.

To clarify what effects diffusion and consumption has on the system, we will vary α.



When α is small, diffusion dominates the system; decrease of attractants due to reaction diminishes. When $\alpha = 0$, the attractant profile in the left region is constant due to the no flux condition; the profile in the right region is linear, just like in a simple diffusion problem. When α is large, consumption dominates and every source of attractant will be consumed immediately.

This is an example of how we can gain insights into the physics of the system with only the Green's function. The final profile will vary depending on the non-homogeneous term, but we know the general concepts of the attractant behavior

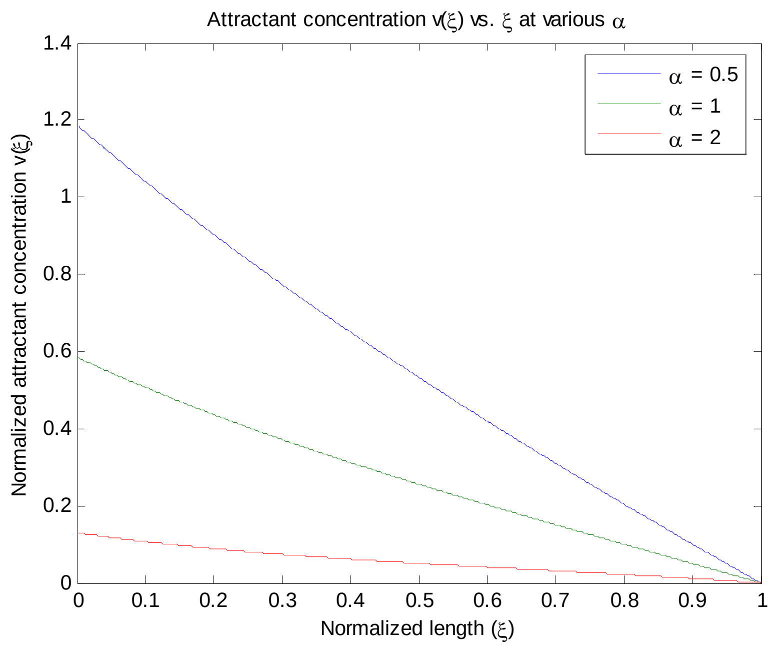
Specific example where $f(\eta) = e^{-\eta}$

Then, we can actually evaluate Eq. (2). Since there are two regions of the Green's function, we need to split up the integral at ξ.

$$v(\xi) = \int_0^{\xi} G(\xi, z) e^{-z} dz + \int_{\xi}^{1} G(\xi, z) e^{-z} dz$$

After many pages of math…

$$v(\xi) = \frac{1}{(\alpha^2 + 1)\cosh\alpha} \left\{ e^{-\xi} \cosh\alpha - \frac{\cosh(\alpha\xi)}{e} + \frac{\sinh[\alpha(\xi - 1)]}{\alpha} \right\}$$



Attractant concentration v(ξ) vs. ξ at various α
Normalized attractant concentration v(ξ)
Normalized length (ξ)
α = 0.5
α = 1
α = 2
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1