Model for bacteria, chemoattractant, and leukocytes in (periodontal) tissue. 9/29

Ended with version of bacterial density conservation equation

scaled bact. density: $\frac{\partial u}{\partial \tau} = u(1-\theta_w)$

θ_w: scaled leukocyte density

because $\Delta_b \equiv \frac{D_b}{k_g L^2} \ll 1$

so, if we know $w(\xi, \tau)$, we can determine $u(\xi, \tau)$

Thus, we'd like to see what governs $w(\xi, \tau)$.

We already know that it will critically depend on $v(\xi, \tau)$ (scaled attractant conc.)

scaled attractant conc:

$$\frac{\partial v}{\partial \tau} = \underbrace{\left[\frac{D_a}{k_g L^2}\right]}_{\text{diffusion}} \frac{\partial^2 v}{\partial \xi^2} + \overbrace{\left[\frac{k_p b_i}{k_g a^*}\right] u}^{\text{production by bact}} - \left[\frac{k_d}{k_m k_g}\right] \frac{v}{1+\left[\frac{a^*}{k_m}\right] v} - \left[\frac{k_u C_{blood}}{k_g}\right] w v$$

or choose $a^* = \frac{k_p b_i}{k_g}$?

choose $a^* = k_m$?

$$\frac{\partial v}{\partial \tau} = \Delta_a \frac{\partial^2 v}{\partial \xi^2} + u - \gamma \frac{v}{1+Kv} - \eta w u$$

$\Delta_a = \frac{D_a}{k_g C^2}$, $\gamma = \frac{k_d}{k_g K_m}$, $K = \frac{k_p b_i}{k_g k_m}$, $\eta = \frac{k_u C_{blood}}{k_g}$

$D_a \sim 3\times10^{-6}\ cm^2/s$

$k_g \sim 3\times10^{-1}\ hr^{-1}$

$L \sim 0.5\ mm$

$\Rightarrow \Delta_a \sim 10$

$\Rightarrow$ attractant diffusion is fast, so attractant dynamics might be reasonably approximated as steady state ie. $\frac{\partial v}{\partial \tau} = 0$

So, our problem can be reduced to $0 = \Delta_a \frac{\partial^2 v}{\partial \xi^2} + u - \gamma \frac{v}{1+kv} - \eta w v$

u dynamics (slow) $\xrightarrow{\text{fast}}$ v dynamics $\rightarrow$ w dynamics (full τ, ξ PDE) $\rightarrow$ u dynamics

τ- ODE w/ no spatial derivative

ξ-ODE w/ no temporal derivative

We'll also assume here that $\eta \ll 1$, so cellular update of attractant can be neglected and also assume $k \ll 1$

$$\Rightarrow \boxed{\Delta_a \frac{\partial^2 v}{\partial \xi^2} - \gamma v} = \boxed{-u(\xi, \tau)}$$

"operator"

attractant diffusion; attractant proteolytic degredation; attractant production by bacteria

"inhomogeneous forcing function"

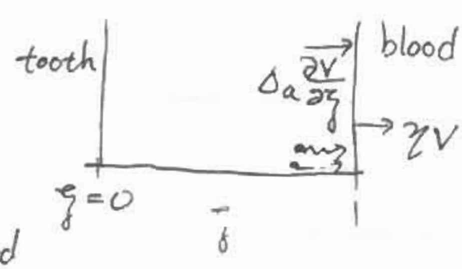
B.C. in scaled form
$$\left.\begin{array}{l} \xi = 0 \\ \frac{\partial v}{\partial \xi} = 0 \end{array}\right\} \text{(tooth surface)}$$

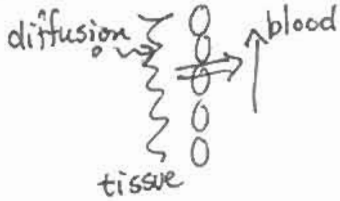
$$\left.\begin{array}{l} \xi = 1 \\ \Delta_a \frac{\partial v}{\partial \xi} + \eta v = 0 \end{array}\right\} \text{(vasculatare)}$$

$\Delta_a = \frac{D_a}{k_g L^2}$, $\eta = \frac{h_a}{k_g L}$

$\xi = 1$ $\quad 0 = \frac{\partial v}{\partial \xi} + \left[\frac{\eta}{\Delta_a}\right] v$, $\quad \frac{\eta}{\Delta_a} \approx \frac{h_a L}{D_a}$

What transport mechanism(s) is/are represented by transfer coefficient?





blood vessel
endothelial cells

$h_a[a - \cancel{a_{blood}}] \approx 0$

Limiting cases for this boundary condition:

if $\frac{\eta}{D_a} \to 0 \Rightarrow 0 = \frac{\partial v}{\partial \xi}$ if transfer from boundary to blood is limited

$\left(\frac{h_a L}{D_a} \ll 1\right)$ on the other hand, if $\frac{\eta}{\Delta_a} \to \infty$

$\Downarrow$

$\left(\frac{h_a L}{D_a} \gg 1\right)$ $0 = v$

if diffusion through tissue to boundary is limiting

Problem @ hand can be written as an inhomogeneous linear operator equation

$-\mathcal{L} v = f$ ← forcing function $(-u)$

v ← dependent variable

$\mathcal{L}$ ↑ linear operator

$\left(-\Delta_a \frac{\partial^2}{\partial \xi^2} - \gamma\right)$

on domain $0 \le \xi < 1$ w/ BC. $\xi = 0$

$$\frac{dv}{\partial \xi} = 0$$

linear & homogeneous

$$\xi = 1 \quad \Delta_a \frac{\partial v}{\partial \xi} + \eta v = 0$$

Green's Functions

- Solution to $-\mathcal{L} v = f$ $\quad \xi_1 \le \xi \le \xi_2$

w/ homogeneous BC can be written as

$$v = \int_{\xi_1}^{\xi_2} G(\xi, z)\,[f(z)]\,dz$$

z ↑ dummy variable

where G is the Green's Function associated w/ operator $\mathcal{L}$ & the BC.

- it's the solution to $\mathcal{L}_\xi G = \delta(z)$ w/ BC.

$\delta(z)$ ↖ delta function

where $\delta(z) = \begin{cases} 0 & \xi \neq z \\ \infty & \xi = z \end{cases}$ & $\int_{\xi_1}^{\xi_2} \delta(\xi)\,d\xi = 1$

for more details, see handout + web posting

Ritger & Rose (calculational details) — handout

Stakgold (intuitive concepts) — web posting

Bottomline result + simple example here

for class of operators of the form

$$\mathcal{L}y = a_0(\xi)\frac{d^2y}{d\xi^2} + a_1(\xi)\frac{dy}{d\xi} + a_2(\xi)y \quad \text{w/ homogeneous B.C.}$$

the

$$G(\xi, z) = \begin{cases} \dfrac{y_2(z)\,y_1(\xi)}{g(z)} & \xi \leq z \\[2ex] \dfrac{y_1(z)\,y_2(\xi)}{g(z)} & \xi \geq z \end{cases}$$

where y_1 is a solution to $\mathcal{L}y_1 = 0$ satisfying one BC

& y_2 is a solution to $\mathcal{L}y_2 = 0$ satisfying other BC

$$g(z) = a_0(z)\left[y_1(z)\frac{dy_2}{d\xi}\bigg|_z - y_2(z)\frac{dy_1}{d\xi}\bigg|_z\right]$$

$a_0(z), a_1(z), a_2(z)$ have to be integrable.

Illustrate w/ basic problem $-\dfrac{d^2v}{d\xi^2} = f(\xi) \quad 0 \leq \xi \leq 1$

BC. $\xi = 0,\ v = 0,\ \xi = 1,\ v = 1$

$a_0 = -1,\ a_1 = 0,\ a_2 = 0$, general solution to $\mathcal{L}y = 0$ is $y = B_1 + B_2\xi$

y_1 satisfy $\xi = 0$ BC $\Rightarrow y_1 = \xi$

y_2 satisfy $\xi = 1$ BC $\Rightarrow y_2 = 1 - \xi$

$$\Rightarrow q(z)=1, \quad \text{so} \quad G(\xi, z) = \begin{cases} (1-z)\,\xi & \xi \le z \\ z(1-\xi) & \xi \geq z \end{cases}$$

Let's plot Green's Function for this $\mathcal{L}$ + BC