Mechanical Subsystem, continued — L17 — 11/10/04

mechanical force balance — $F = ma \Rightarrow$ useful balance eqn for biological fluid situations, incompressible fluid, linear stress-strain, Newtonion viscosity flow/law.

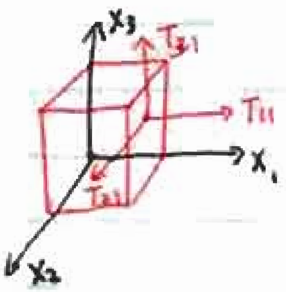
$\boxed{\text{Navier-Stokes Eqn}}$

fluid velocity — electric field forces — $\boxed{\nabla \cdot \underline{T}}$

$$\rho \frac{D\underline{v}}{Dt} = \rho g + \rho_e E - \nabla p + \mu \nabla^2 \underline{v}$$

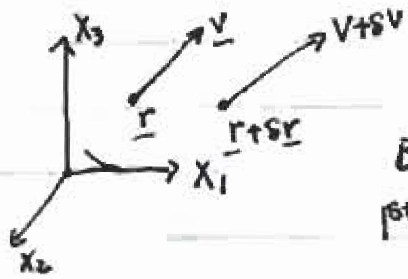
fluid density — gravity — pressure gradient — viscous stress

arise from surface force tensor, $\underline{T}$



$$\underline{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

T_{ij} = force per area in i^{th} direction on surface whose normal is in j direction



unit vector

let $\underline{v} = v_1 \underline{x}_1 + v_2 \underline{x}_2 + v_3 \underline{x}_3$

By Taylor series expansion, truncating @ 1^{st} order $\delta v = \underline{D}\,\delta r + O(\delta r^2)$

fluid deformation

$$\underline{\delta v} = \begin{bmatrix} \delta v_1 \\ \delta v_2 \\ \delta v_3 \end{bmatrix} \qquad \underline{\delta r} = \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix}$$

$$\underline{D} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

$\underline{\delta v} = \underline{D}\,\underline{\delta r}$ or $\delta v_i = \frac{\partial v_i}{\partial x_j} \delta x_j$

"Einstein Notation for summation"

Decompose $\underline{\underline{D}}$ into symmetric and anti-symmetric parts

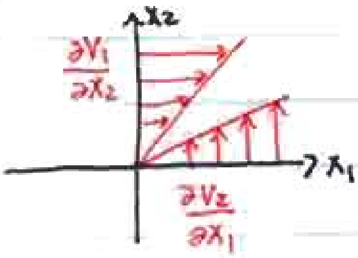
$$\underline{\underline{D}} = \underline{\underline{e}} + \underline{\underline{\eta}}$$

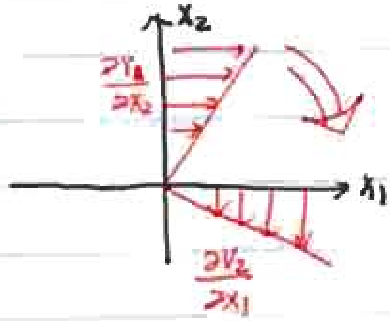
$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$ shear

$e_{ij} = \underline{\underline{\varepsilon}}$

$\eta_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$ rotation

$\eta_{ij} \Leftarrow \underline{\underline{\eta}}$





Newtonian fluid:

$$\underline{\underline{T}} = \underline{\underline{\underline{\underline{C}}}}\ \underline{\underline{\varepsilon}}$$

$\underline{\underline{T}}$: surface stress tensor

$\underline{\underline{\underline{\underline{C}}}}$: $3\times3\times3\times3 = 81$ coefficients

$\underline{\underline{\varepsilon}}$: strain rate tensor

In an isotropic fluid, 81 reduces to 2. So, $T_{ij} = 2\mu e_{ij} + (\lambda - \frac{2}{3}\mu)\delta_{ij} e_{kk}$

T_{ij}: $\underline{\underline{T}}$ (3x3)

μ: 1st Lamé coefficient (viscosity)

$(\lambda - \frac{2}{3}\mu)$: 2nd Lamé coefficient

δ_{ij}: Kronecker delta $\begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

In an incompressible fluid (essentially all of biology)

$$T_{ij} = 2\mu e_{ij} - p\delta_{ij} \leftarrow \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

T_{ij}: $\underline{\underline{T}}$

μ: viscosity

e_{ij}: fluid strain rate

p: pressure

Scaling argument:

let $\underset{\sim}{\omega} = \frac{v}{V_{max}}$, $\xi = \frac{r}{L}$, $\tau = \frac{t}{(L/V_{max})}$, then $\frac{\rho \frac{Dv}{Dt}}{\mu \nabla_r^2 v} - \boxed{\left(\rho \frac{V_{max} L}{\mu}\right)} \frac{\frac{D\underline{v}}{D\tau}}{\nabla_\xi^2 \underline{w}}$

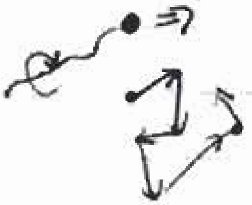
V_{max}, L: presume

Reynold's Number

In most (though not all), biological fluid systems:

$Re \ll 1$

Example - bacterial motility in water



$L \sim 1\,\mu m$, $V \sim 10\,\mu m/sec$

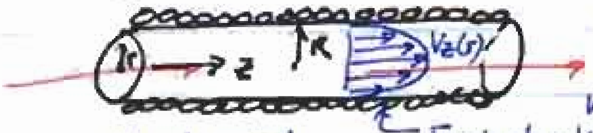
$\rho \sim 1\,g/ml$

$\mu \sim 1\,cp = 10^{-3}\,kg/m \cdot sec$

$Re = 10^{-5}$ ⇒ good approximation in most biology problems is

$$\rho \frac{D\underline{v}}{Dt} \to 0 \Rightarrow \boxed{0 = \rho \underline{g} + \underbrace{\rho_e E}_{\text{often negligible}} - \nabla p + \mu \nabla^2 \underline{v}} \text{ Stokes Eqn}$$

Quick + easy example #1: What is shear force on ECs?
↑ endothelial cells



steady state (no change w/ time), fully developed (no change w/ direction of flow)

Stokes Eqn: $0 = \frac{\mu}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) - \frac{\partial p}{\partial z}$ ← pressure drop in axial direction

Non-Newtonian behavior of blood due to high protein conc & cell densities - exacerbated in domains ~ cell dimension.

$$v_z(r) = 2\bar{v}\left(1 - \left[\frac{r}{R}\right]^2\right) \qquad \bar{v} = \frac{\int_0^R v_z(r)\, 2\pi r\, dr}{\int_0^R 2\pi r\, dr} = \frac{R^2}{8\mu}\left(-\frac{dp}{dz}\right)$$

↑ mean velocity

Volumetric flow rate $Q = \bar{V} \cdot \pi R^2 = \frac{\pi R^4}{8\mu}\left(-\frac{dp}{dz}\right)$

viscous fluid stress on vessel wall (ECs):

$$\tau_{zr} = -\mu \frac{dV_z}{dr}\Big|_{r=R} = -\frac{4\mu\bar{V}}{R}$$

Example numbers -

Venule $R \sim 20\,\mu m$

$V \sim 200\,\mu m/s$

$\mu \sim 3\,cp$ ($\sim 3 \times 10^{-3}$ kg/m sec)

blood, thicker than water

From Example 2.

Overall force on bacterium

$$F_{\text{"drag"}} = -2\pi R^2 \int_0^\pi [p(R,\theta)\cos\theta + \tau(R,\theta)\sin\theta]\, d\theta$$

$2\pi\mu V_{max} R$ (pressure) $\quad 4\pi\mu V_{max} R$ (shear).

$= 6\pi\mu V_{max} R$

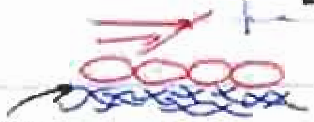
Example. $R \sim 1\,\mu m$

$V_{max} \sim 10\,\mu m/sec$

$\mu \sim 10^{-3}$ kg/m.s

$F_{drag} \sim 200\,pN$.

$\tau_{zr} \sim 0.1\,N/m^2$, $0.1\,pN/\mu m^2$



$\sim 10^4 - 10^5$ adhesion receptor / ECM bonds per cell (10^3 pN / cell)

each ~ 1 pN

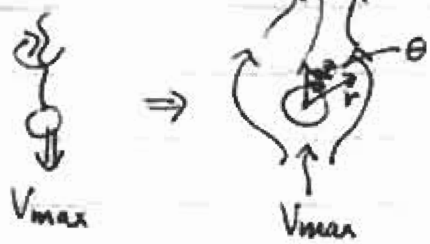
cell $\sim 10\,\mu m$ radius $\Rightarrow \sim 100\,\mu m^2$ surface area

Stopping distance?

$ma = 6\pi\mu V_{max} R$

$V_{max} = 0 \quad 10^{-2}\,\mu m$

Example #2:



Stoke's Eqn

$$0 = \nabla p + \mu \nabla^2_{r\theta} \underline{v}$$

Viscous drag force

$$\tau_{r\theta}(R,\theta) = -\frac{3}{2}\frac{\mu V_{max}}{R}\sin\theta$$

Pressure force: $p(r,\theta) = -\frac{3}{2}\frac{\mu V_{max}}{R}\left(\frac{R}{r}\right)^2\cos\theta$

B.C. $r \to \infty$, all θ, $V(r,\theta) \to V_{max}$

$r \to \infty$, all θ, $p \to p_\infty$

See example 7.4-2 Deen text: $V_r(r,\theta) = V_{max}\cos\theta\left[1 - \frac{3}{2}\left(\frac{R}{r}\right) + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]$

$V_\theta(r,\theta) = -V_{max}\sin\theta\left[1 - \frac{3}{4}\left(\frac{R}{r}\right) - \frac{1}{4}\left(\frac{R}{r}\right)^3\right]$

Overall force: $F_{\text{"drag"}} = -2\pi R^2 \int_0^\pi [p(R,\theta)$