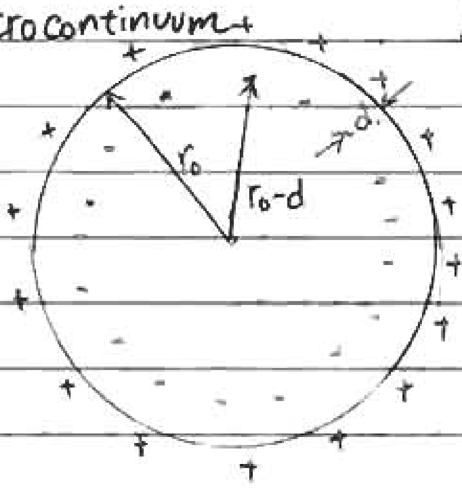
Solution

6.2.1 Microcontinuum



Helmholtz double layer model.

Approximations and Assumptions.

- $d = 1/k$
- σ (conductivity) is uniform across cross section
- $\delta \ll d \ll r_0$ where $r = r_0 - \delta$ is "slip plane"
- $\Phi(r) \approx 0,\ r \le (r_0-d)$
  $\Phi(r) \approx \frac{\xi[r-(r_0-d)]}{d},\ (r_0-d) \le r \le r_0$
- $\sigma_d \approx \left(\frac{\varepsilon}{d}\right)\xi$ where σ_d is surface charge density
- $\rho_u \approx -\sigma_d u_0(r-(r_0-d))$; for Helmholtz double layer model, ρ_u is approx. as impulse surface charge @ $r = r_0 - d$.

EQN 4: $i_p = \int_0^{r_0} 2\pi r \rho_u(r) u_z(r)\,dr - \int_0^{r_0} 2\pi r \sigma(r) \left\{\frac{[\![\psi_p]\!]}{l_p}\right\} dr$.

$[\![\Pi_p]\!]$ comp. of EQN 4 : $2\pi \int_0^{r_0} \rho_u r \frac{[r^2-(r_0-\delta)^2]}{4 l_p \eta}\,dr\,[\![\pi_p]\!]$

$$= \frac{-\pi [\![\pi_p]\!]}{2 l_p \eta} \int_0^{r_0} r \left(\frac{\varepsilon}{d}\right)\xi\, u_0(r-(r_0-d))[r^2-(r_0-\delta)^2]\,dr$$

$$= \frac{-\pi\varepsilon\xi [\![\Pi_p]\!]}{l_p \eta d}\left[(r_0-d)^2-(r_0-\delta)^2\right](r_0-d)$$

$$= \frac{-\pi\varepsilon\xi [\![\Pi_p]\!]}{l_p \eta d}\left[\cancel{r_0^2}-2r_0 d+\cancel{d^2}-\cancel{r_0^2}+\cancel{2r_0\delta}-\cancel{\delta^2}\right] r_0$$

$$= \frac{-\pi \varepsilon \xi [\![\pi_p]\!]}{\cancel{2}\, l_p \eta \cancel{d}} (-\cancel{2} r_0^2 \cancel{d}) \Rightarrow \boxed{\frac{\pi \varepsilon \xi r_0^2}{\eta l_p} = g_{21}} \leftarrow$$

correspond to g21 in EQN 10 below

EQN 10: $$\begin{bmatrix} Q_p \\ i_p \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} [\![\pi_p]\!] \\ [\![\psi_p]\!] \end{bmatrix}$$

$[\![\psi_p]\!]$ comp. of i_p: $$\left\{2\pi \int_0^{r_0} r \rho_u(r) \left\{\frac{\varepsilon[\xi - \Phi(r)]}{l_p \eta}\right\} dr\; 2\pi \int_0^{r_0} \frac{r \sigma(r)}{l_p}\, dr\right\} [\![\psi_p]\!]$$

$$= \left\{\frac{-2\pi}{l_p \eta} \frac{\varepsilon^2}{d} \xi \left[(\overbrace{r_0 - d}^{\sim r_0}) [\xi - \cancel{\Phi(r_0 - d)}]^{0}\right] - \frac{2\pi\sigma}{l_p} \frac{r^2}{\cancel{2}}\Big]_0^{r_0}\right\} [\![\psi_p]\!]$$

$$\Rightarrow \boxed{\left\{\frac{-2\pi \varepsilon^2 \xi^2 r_0}{l_p d \eta} - \frac{\pi \sigma r_0^2}{l_p}\right\} = g_{22}} \leftarrow g_{22} \text{ in EQN 10.}$$

$$Q_p = \int_0^{r_0} 2\pi r\, u_z(r)\, dr$$

$$= 2\pi \int_0^{r_0} r \frac{[r^2 - (\overbrace{r_0 - \delta}^{\sim r_0})^2]}{4 l_p \eta} dr\, [\![\pi_p]\!] + 2\pi \int_0^{r_0} \frac{r \varepsilon [\xi - \Phi(r)]}{l_p \eta} dr\, [\![\psi_p]\!]$$

$[\![\pi_p]\!]$ Comp of Q_p: $$= \frac{\pi}{2 l_p \eta} \left[\frac{r^4}{4} - \frac{r_0^2 r^2}{2}\right]_0^{r_0} [\![\pi_p]\!] = \frac{\pi}{2 l_p \eta} \left[\frac{r_0^4}{4} - \frac{r_0^4}{2}\right] = \boxed{\frac{-r_0^4 \pi}{8 l_p \eta} = g_{11}}$$

g_{11} in EQN 10

$[\![\psi_p]\!]$ comp. of Q_p:
$$\left\{\frac{2\pi \varepsilon \xi}{l_p \eta} \frac{r^2}{\cancel{2}}\Big]_0^{r_0} - \frac{2\pi \varepsilon}{l_p \eta} \left[\cancel{\int_0^{r_0 - d} r\, 0\, dr}^{\,0} + \int_{r_0 - d}^{r_0} \frac{r \xi [r - (\overbrace{r_0 - d}^{r_0})]}{d}\right]\right\} [\![\psi_p]\!]$$

$$= \left\{\frac{\pi \varepsilon \xi r_0^2}{l_p \eta} \quad \frac{2\pi \varepsilon \xi}{l_p \eta d} \left[\frac{r^3}{3} - \frac{r_0 r^2}{2}\right]_{r_0 - d}^{r_0}\right\} [\![\psi_p]\!]$$

$$= \left\{ \frac{\pi \varepsilon \xi r_0^2}{l_p \eta} - \frac{2\pi \varepsilon \xi}{l_p \eta d}\left[\frac{\cancel{r_0^3}}{\cancel{3}} - \frac{\cancel{r_0^3}}{\cancel{2}} - \frac{\overbrace{\cancel{(r_0-d)^3}}^{r_0}}{\cancel{3}} + \frac{\overbrace{(r_0-d)^3}^{r_0}}{\cancel{2}} \right] \right\} [\![\psi_p]\!]$$

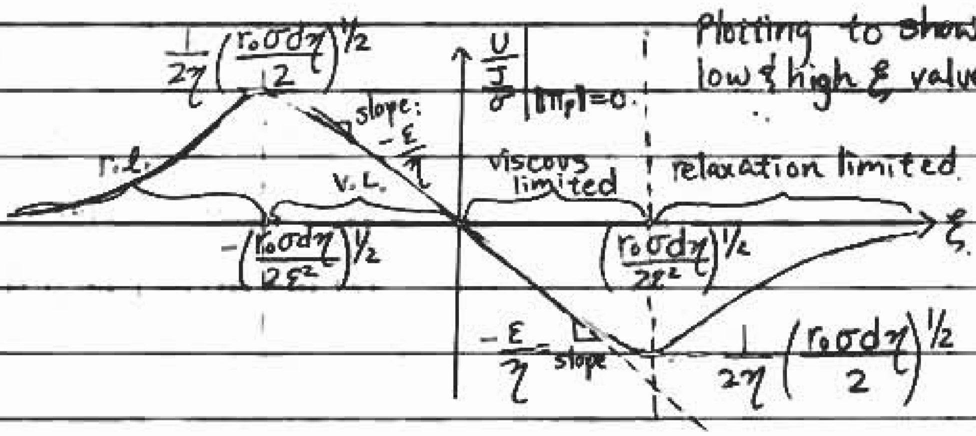
$$\Rightarrow \boxed{\frac{\pi \varepsilon \xi r_0^2}{l_p \eta} = g_{12}} \text{— } g_{12} \text{ in EQN 10.}$$

$$\begin{bmatrix} Q_p \\ i_p \end{bmatrix} = \begin{bmatrix} -\frac{r_0^4 \pi}{8 l_p \eta} & \frac{\pi \varepsilon \xi r_0^2}{\eta l_p} \\ \frac{\pi \varepsilon \xi r_0^2}{\eta l_p} & -\left(\frac{2\pi r_0 \varepsilon^2 \xi^2}{\eta d l_p} + \frac{\pi \sigma r_0^2}{l_p} \right) \end{bmatrix} = \begin{bmatrix} [\![\pi_p]\!] \\ [\![\psi_p]\!] \end{bmatrix} \checkmark$$

b) $$\left. \frac{Q_p}{i_p} \right|_{[\![\pi_p]\!]=0} = \frac{g_{12}}{g_{22}} = \frac{-\frac{\cancel{\pi} \varepsilon \xi r_0^{\cancel{2}}}{\eta \cancel{l_p}}}{-\left(\frac{2\cancel{\pi} \cancel{r_0} \varepsilon^2 \xi^2}{\eta d \cancel{l_p}} + \frac{\cancel{\pi} \sigma r_0^2}{\cancel{l_p}}\right)} = \boxed{\frac{-\varepsilon \xi r_0 d}{2\varepsilon^2 \xi^2 + \sigma r_0 \eta d}}$$

$$\left. \frac{U}{\frac{J}{\sigma}} \right|_{[\![\pi_p]\!]=0} = \frac{-\frac{\varepsilon \xi}{\eta \cancel{l_p}}}{\frac{2\varepsilon^2 \xi^2}{\eta d \cancel{l_p} \sigma r_0} + \frac{1}{\cancel{l_p}}} = \boxed{\frac{-\varepsilon \xi}{\frac{2\varepsilon^2 \xi^2}{d \sigma r_0} + \eta}}$$

c)



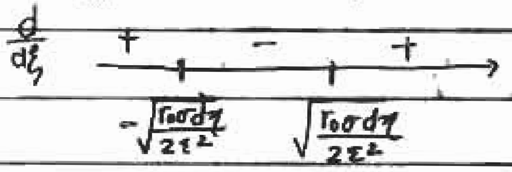
Plotting to show dependence low & high ξ values on $\left.\frac{U}{J/\sigma}\right|_{[\![\pi_p]\!]=0}$

$$\frac{d}{d\xi}\left(\frac{u}{J/\sigma}\bigg|_{[\![\pi_p]\!]=0}\right)=\frac{-\frac{\xi}{\eta}\left(\frac{2\varepsilon^2\xi^2}{r_0\sigma d\eta}+1\right)+\frac{\varepsilon\xi}{\eta}\left(\frac{4\varepsilon^2\xi}{r_0\sigma d\eta}\right)}{\left(1+\frac{2\varepsilon^2\xi^2}{r_0\sigma d\eta}\right)^2}=0$$

$$-\frac{2\varepsilon^3\xi^2}{r_0\sigma d\eta^2}=\frac{\varepsilon}{\eta}\Rightarrow\xi^2=\frac{\cancel{\varepsilon}}{\cancel{\eta}}\cdot\frac{r_0\sigma d\cancel{\eta^2}}{2\varepsilon^{3/2}}$$

$$\xi_{roots}=\pm\sqrt{\frac{r_0\sigma d\eta}{2\varepsilon^2}}$$ where derivative $=0$.

Derivative Test:



$\xi\approx\frac{d}{\varepsilon}\sigma_d \Rightarrow$ ~~Substitute~~ ξ with σ_d.

$$\frac{u}{J/\sigma}\bigg|_{[\![\pi_p]\!]=0}=\frac{-\cancel{8}\frac{d}{\cancel{8}}\sigma_d}{\frac{2\cancel{\varepsilon^2}d^2\sigma_d^2}{\cancel{\varepsilon^2}\cancel{d}\sigma r_0}+\eta}=\boxed{\frac{-d\sigma_d}{\frac{2\sigma_d^2 d}{r_0\sigma}+\eta}}$$

If $\frac{2\sigma_d^2 d}{r_0\sigma}\ll 1$, $\frac{u}{J/\sigma}\bigg|_{[\![\pi_p]\!]=0}\simeq\frac{-d\sigma_d}{\eta}$ ← since this only depends on η, a viscous term, ∴ $\frac{u}{J/\sigma}$ becomes $\boxed{\text{viscous limited.}}$

↑ This term in denominator can be neglected.

If $\frac{2\sigma_d^2 d}{r_0\sigma}\gg 1$, $\frac{u}{\frac{J}{\sigma}}\bigg|_{[\![\pi_p]\!]=0}\sim\frac{-1}{\frac{2\sigma d}{r_0\sigma}}$ ↖ since this term neglects η, & dominiated by σ, $\frac{u}{J/\sigma}$ become $\boxed{\text{relaxation}}$ or conductivity $\boxed{\text{limited.}}$

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d) $\left.\dfrac{u}{J/\sigma}\right|_{[\![\pi_p]\!]=0} = \boxed{\dfrac{-\varepsilon\zeta}{\eta}}$ when $\sigma \to$ high ①

$$\left.\frac{\frac{Q_p}{\pi r_0^2}}{\frac{-[\![\psi_p]\!]}{l_p}}\right|_{[\![\pi_p]\!]=0} = \frac{\left[\cancelto{0}{\frac{-\pi r_0^{\cancel{4}2}}{8\eta l_p}[\![\pi_p]\!]} + \frac{\cancel{\pi r_0^2}\varepsilon\zeta}{\eta l_p}[\![\psi_p]\!]\right]\frac{\cancel{l_p}}{\cancel{\pi r_0^2}}}{-[\![\psi_p]\!]} = \boxed{\frac{-\varepsilon\zeta}{\eta}} \checkmark \quad ②$$

$$② = -\left(\frac{l_p}{\pi r_0^2}\right)g_{12} = \frac{-l_p}{\pi r_0^2}\cdot\frac{\pi r_0^2\varepsilon\zeta}{\eta l_p} = \frac{-\varepsilon\zeta}{\eta} = ①. \checkmark.$$

Also $\boxed{J = \underbrace{\sigma E_z}_{\text{electric Component}} + \underbrace{\rho_e u}_{\text{convection term.}}}$ from Eq 10 in AJG Text pg 27.

$$\rho_{tot} = -\sigma_d \cdot \frac{2\pi r_0 l_p}{\pi r_0^2 l_p} = -\sigma_d \cdot \frac{2}{r_0}$$

(σ_d is surface charge density, but ρ_{tot} needs to be volume charge density, so σ_d needs to multiply by SA/vol ratio).
(ρ_{tot} is negative because we consider negative charge on inside of channel wall)

$$E_z = \frac{J}{\sigma} + \frac{\sigma_d 2\pi r_0 l_p u}{\sigma\pi r_0^2 l_p} = \frac{-[\![\psi_p]\!]}{l_p}.$$

$$= \frac{J}{\sigma}\left(1 + \frac{\sigma_d 2\pi r_0 l_p u}{\pi r_0^2 l_p J}\right) = \underbrace{\frac{J}{\sigma}\left(1 + \frac{2\sigma_d u}{r_0 J}\right)}_{} = \frac{-[\![\psi_p]\!]}{l_p}.$$

substitute above expression into ②

$$\frac{u}{E_z} = \frac{u}{\frac{J}{\sigma} + \frac{2\sigma_d u}{\sigma r_0}} = \frac{-\varepsilon\zeta}{\eta} \Rightarrow u = \frac{-\varepsilon\zeta}{\eta}\left[\frac{J}{\sigma} + \frac{2\sigma_d u}{\sigma r_0}\right]$$

$$u\left(1 + \frac{2\sigma_d\varepsilon\zeta}{\sigma r_0 \eta}\right) = -\frac{J}{\sigma}\frac{\varepsilon\zeta}{\eta}$$

$$\boxed{\therefore \frac{u}{J/\sigma} = \frac{-\varepsilon\zeta}{\eta}\left(1 + \frac{2\sigma_d\varepsilon\zeta}{\sigma r_0\eta}\right)^{-1}}$$